

NOETHER'S PROBLEM FOR CENTRAL EXTENSIONS OF METACYCLIC p -GROUPS

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ABSTRACT. Let K be a field and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. Noether's problem then asks whether $K(G)$ is rational over K . In [M. Kang, Noether's problem for metacyclic p -groups, Adv. Math. 203(2005), 554-567], Kang proves the rationality of $K(G)$ over K if G is any metacyclic p -group and K is any field containing enough roots of unity. In this paper, we give a positive answer to the Noether's problem for all central group extensions of the general metacyclic p -group, provided that K is infinite and it contains sufficient roots of unity.

1. INTRODUCTION

Let K be a field and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. *Noether's problem* then asks whether $K(G)$ is rational (= purely transcendental) over K . It is related to the inverse Galois problem, to the existence of generic G -Galois extensions over k , and to the existence of versal G -torsors over k -rational field extensions [Sw, Sa1, GMS, 33.1, p.86].

The following well-known theorem gives a positive answer to the Noether's problem for abelian groups.

Theorem 1.1. (Fischer [Sw, Theorem 6.1]) *Let G be a finite abelian group of exponent e . Assume that (i) either $\text{char } K = 0$ or $\text{char } K > 0$ with $\text{char } K \nmid e$, and (ii) K contains a primitive e -th root of unity. Then $K(G)$ is rational over K .*

Swan's paper [Sw] also gives a survey of many results related to the Noether's problem for abelian groups. In the same time, just a handful of results about Noether's

Date: September 16, 2011.

1991 Mathematics Subject Classification. 12F12, 13A50, 11R32, 14E08.

Key words and phrases. Noether's problem, the rationality problem, metacyclic p -groups.

This work is partially supported by a project No RD-05-156/25.02.2011 of Shumen University.

problem are obtained when the groups are nonabelian. The reader is referred to [CK, Ka1, HuK, Ka2] for previous results of Noether's problem for p -groups.

We state now the result obtained recently by Kang [Ka1] about the Noether's problem for metacyclic p -groups:

Theorem 1.2. (Kang [Ka1, Theorem 1.5]) *Let G be a metacyclic p -group with exponent p^e , and let K be any field such that (i) $\text{char } K = p$, or (ii) $\text{char } K \neq p$ and K contains a primitive p^e -th root of unity. Then $K(G)$ is rational over K .*

It is still an open question whether the above result could be extended for all series of 2-generator p -groups or meta-abelian groups, that have quotient groups isomorphic to the metacyclic p -groups. However, we should not "over-generalize" Theorem 1.2, because Saltman proves the following result.

Theorem 1.3. (Saltman [Sa2]) *For any prime number p and for any field K with $\text{char } K \neq p$ (in particular, K may be an algebraically closed field), there is a meta-abelian p -group G of order p^9 such that $K(G)$ is not rational over K .*

Among the known results of Noether's problem for non-abelian p -groups, assumptions on the existence of "enough" roots of unity always arose. In fact, even when G is a non-abelian p -group of order p^3 where p is an odd prime number, it is not known how to find a necessary and sufficient condition such as $\mathbb{Q}(G)$ is rational over \mathbb{Q} (see [Ka3]). Thus it will be desirable if we can weaken the assumptions on the existence of roots of unity.

The purpose of this paper is to extend Theorem 1.2 for all central p -extensions of the general metacyclic p -group. However, some additional assumptions will appear in the statements of our results so that we guarantee the existence of the groups we are going to consider.

Let G be any metacyclic p -group generated by two elements σ and τ with relations $\sigma^{p^a} = 1, \tau^{p^b} = \sigma^{p^c}$ and $\tau^{-1}\sigma\tau = \sigma^{\varepsilon+\delta p^r}$ where $\varepsilon = 1$ if p is odd, $\varepsilon = \pm 1$ if $p = 2$, $\delta = 0, 1$ and $a, b, c, r \geq 0$ are subject to some restrictions. For the description of these restrictions see e.g. [Ka1, p. 564]. Note that if $\delta = 0$ and $\varepsilon = 1$, then G is abelian group generated by two elements.

Our first main result is the following.

Theorem 1.4. *Let G be an abelian metacyclic p -group generated by two elements σ and τ with relations $\sigma^{p^a} = 1, \tau^{p^b} = 1$ and $\tau^{-1}\sigma\tau = \sigma$. Assume that \tilde{G} is a central extension of G , i.e., we have the following group extension*

$$1 \longrightarrow C \longrightarrow \tilde{G} \longrightarrow G \cong C_{p^a} \times C_{p^b} \longrightarrow 1,$$

where $C \leq Z(\tilde{G})$. Let p^t be the exponent of C , let $a \geq b \geq t$ and let the pre-image of $[\sigma, \tau] = \sigma^{-1}\tau^{-1}\sigma\tau$ in \tilde{G} is of order p^t . Let $e = \max\{a, 2t\}$. Assume that (i) $\text{char}K = p$ or (ii) $\text{char}K \neq p$, K is infinite, and K contains a primitive p^e -th root of unity. Then $K(\tilde{G})$ is rational over K .

Next, we consider the case when G is a nonabelian metacyclic p -group. The second main result of this paper is the following theorem that concerns the central extensions of G .

Theorem 1.5. *Let G be a nonabelian metacyclic p -group generated by two elements σ and τ with relations $\sigma^{p^a} = 1, \tau^{p^b} = \sigma^{p^c}$ and $\tau^{-1}\sigma\tau = \sigma^{\varepsilon+p^r}$, where $1 \leq c \leq a, r \leq \min\{b, c\}$; $\varepsilon = 1$ if p is odd and $\varepsilon = \pm 1$ if $p = 2$. Assume that \tilde{G} is a central extension of the group G by the cyclic group C_{p^t} , i.e., we have the following group extension*

$$1 \longrightarrow C_{p^t} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

where $C_{p^t} \leq Z(\tilde{G})$. Let $a \geq t, b \geq t$ and let the pre-image of $\sigma^{-(k-1)}[\sigma, \tau] = \sigma^{-k}\tau^{-1}\sigma\tau$ in \tilde{G} is of order p^t . Let $p^m = \exp(\tilde{G})$ and $e = \max\{m, r + t\}$. Assume that (i) $\text{char}K = p$ or (ii) $\text{char}K \neq p$, K is infinite, and K contains a primitive p^e -th root of unity. Then $K(\tilde{G})$ is rational over K .

We can generalize Theorem 1.5 for all central extensions (i.e., not necessarily cyclic extensions), with the expense of a stronger requirement for the root of unity. One can see from our proof that this new condition can be weakened in most of the cases we consider. However, we prefer for simplicity to refrain from giving these details in the statement of the following result.

Corollary 1.6. *Let G be a nonabelian metacyclic p -group generated by two elements σ and τ with relations $\sigma^{p^a} = 1, \tau^{p^b} = \sigma^{p^c}$ and $\tau^{-1}\sigma\tau = \sigma^{\varepsilon+p^r}$, where $\varepsilon = 1$ if p is odd and $\varepsilon = \pm 1$ if $p = 2$. Assume that \tilde{G} is a central extension of the group G , i.e., we*

have the following group extension

$$1 \longrightarrow C \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

where $C \leq Z(\tilde{G})$. Let p^t be the exponent of C , let $a \geq t, b \geq t$ and let the pre-image of $\sigma^{-(k-1)}[\sigma, \tau] = \sigma^{-k}\tau^{-1}\sigma\tau$ in \tilde{G} is of order p^t . Let $e = a + b + t - c$. Assume that (i) $\text{char}K = p$ or (ii) $\text{char}K \neq p$, K is infinite, and K contains a primitive p^e -th root of unity. Then $K(\tilde{G})$ is rational over K .

If $r \geq t$, we are able to weaken the condition for the roots of unity in Theorem 1.5.

Theorem 1.7. *Let G be a nonabelian metacyclic p -group generated by two elements σ and τ with relations $\sigma^{p^a} = 1, \tau^{p^b} = \sigma^{p^c}$ and $\tau^{-1}\sigma\tau = \sigma^{\varepsilon+p^r}$, where $1 \leq c \leq a, r \leq \min\{b, c\}$; $\varepsilon = 1$ if p is odd and $\varepsilon = \pm 1$ if $p = 2$. Assume that \tilde{G} is a central extension of the group G by the cyclic group C_{p^t} , i.e., we have the following group extension*

$$1 \longrightarrow C_{p^t} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

where $C_{p^t} \leq Z(\tilde{G})$. Let $t \leq r$ and let the pre-image of $\sigma^{-(k-1)}[\sigma, \tau] = \sigma^{-k}\tau^{-1}\sigma\tau$ in \tilde{G} is of order p^t . Let $p^m = \exp(\tilde{G})$. Assume that (i) $\text{char}K = p$ or (ii) $\text{char}K \neq p$, K is infinite, and K contains a primitive p^m -th root of unity. Then $K(\tilde{G})$ is rational over K .

We organize this paper as follows. In Section 2 we recall some preliminaries which will be used in the proofs of Theorems 1.4, 1.5, 1.7 and Corollary 1.6. We give the presentations of all cyclic central p -extensions of the metacyclic p -groups in Section 3. We also prove some key results in Sections 2 and 3 that will aid our investigations later in our work. One of these results is Theorem 2.5 which is of interest itself. The proofs of Theorems 1.4, 1.5 and 1.7 are given, respectively, in Sections 4, 5 and 7. In Section 6 the proof of Corollary 1.6 is given.

Standing notations. A field extension L of K is rational over K if L is purely transcendental over K . Recall that $K(G)$ denotes $K(x(g) : g \in G)^G$ where $g \cdot x(h) = x(gh)$ for any $g, h \in G$. A group G is called metacyclic, if G can be generated by two elements σ and τ , and one of them generates a normal subgroup of G . The exponent of a finite group G is $\text{lcm}\{\text{ord}(g) : g \in G\}$ where $\text{ord}(g)$ is the order of g . Two extension

fields L_1 and L_2 of K with G -actions are G -isomorphic if there is an isomorphism $\varphi : L_1 \rightarrow L_2$ over K such that $\varphi(\sigma \cdot u) = \sigma \cdot \varphi(u)$ for any $\sigma \in G$, any $u \in L_1$.

2. GENERALITIES

We list several results which will be used in the sequel.

Theorem 2.1. ([HK, Theorem 1]) *Let G be a finite group acting on $L(x_1, \dots, x_m)$, the rational function field of m variables over a field L such that*

- (i): *for any $\sigma \in G$, $\sigma(L) \subset L$;*
- (ii): *the restriction of the action of G to L is faithful;*
- (iii): *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in \text{GL}_m(L)$ and $B(\sigma)$ is $m \times 1$ matrix over L . Then there exist $z_1, \dots, z_m \in L(x_1, \dots, x_m)$ so that $L(x_1, \dots, x_m)^G = L^G(z_1, \dots, z_m)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq m$.

Theorem 2.2. ([AHK, Theorem 3.1]) *Let G be a finite group acting on $L(x)$, the rational function field of one variable over a field L . Assume that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma x + b_\sigma$ for any $a_\sigma, b_\sigma \in L$ with $a_\sigma \neq 0$. Then $L(x)^G = L^G(z)$ for some $z \in L[x]$.*

Theorem 2.3. ([CK, Theorem 1.7]) *If $\text{char} K = p > 0$ and \tilde{G} is a finite p -group, then $K(G)$ is rational over K .*

Let $\text{Br}(K)$ denote the Brauer group of a field K , and $\text{Br}_N(K)$ its N -torsion subgroup for any $N > 1$. Following Roquette [Ro], if $\gamma = [B] \in \text{Br}(K)$ is the class of a K -central simple algebra B and $m \geq 1$ is a multiple of the index of B , then $F_m(\gamma)$ denotes the m -th Brauer field of γ . Moreover, $F_m(\gamma)/K$ is a regular extension of transcendence degree $m - 1$, which is rational if and only if γ is trivial. The following Theorem was essentially obtained by Saltman in [Sa3, p. 541] and proved in detail by B. Plans [Pl, Prop. 7].

Theorem 2.4. (Saltman [Pl, Proposition 7]) *Let $1 \rightarrow C \rightarrow H \rightarrow G \rightarrow 1$ be a central extension of finite groups, representing an element $\varepsilon \in H^2(G, C)$. Let K be an infinite field and let N denote the exponent of C . Assume that N is prime to the characteristic of K and that K contains μ_N – the group of N -th roots of unity. Let be given a decomposition $C \cong \mu_{N_1} \times \cdots \times \mu_{N_r}$, and let the corresponding isomorphism $H^2(G, C) \cong \bigoplus_i H^2(G, \mu_{N_i})$ map ε to $(\varepsilon)_i$. Let also be given a faithful subrepresentation V of the regular representation of G over K , and let $\gamma_i \in \text{Br}_N(K(V)^G) \subset \text{Br}(K(V)^G)$ be the inflation of ε_i with respect to the isomorphism $G \cong \text{Gal}(K(V)/K(V)^G)$. Then*

$$K(H) \text{ is rational over the } K(V)^G - \text{ free compositum } F_m(\gamma_1) \cdots F_m(\gamma_r),$$

where m denotes the order of G .

The following key result was suggested by Plans [Pl, Proposition 9], but the proof of case 9a, which we need in our paper, is somewhat hard to extract. We will reprove this result for p -group extensions, following a different approach that will allow us to weaken the condition for the roots of unity.

Theorem 2.5. (Plans [Pl, Proposition 9a]) *Let $1 \rightarrow C \rightarrow H \rightarrow G \rightarrow 1$ be a central extension of finite p -groups for a prime p . Let G' (resp. H') be the derived subgroup of G (resp. H), and let p^n (resp. p^e) denote the exponent of C (resp. G/G'). Assume that $H' \cap C = \{1\}$.*

(i) *Let $C = \mu_{p^n}$ and set $m = \max\{i : \mu_{p^i} \cap \mu_{p^n} \neq \{1\}, \mu_{p^i} \leq H/H'\}$. Let K be an infinite field containing the p^m -th roots of unity. Then $K(H)$ is rational over $K(G)$.*

(ii) *Let K be an infinite field containing the p^{n+e} -th roots of unity. Then $K(H)$ is rational over $K(G)$.*

Proof. Let $\varepsilon \in H^2(G, C)$ be the class of $1 \rightarrow C \rightarrow H \rightarrow G \rightarrow 1$. Define $k = K(G) = K(W)^G$, where W denotes the regular representation of G over K . For each $\mu_j \subset K^*$ we have an inflation map $\text{inf} : H^2(G, \mu_j) \rightarrow {}_j\text{Br}(k) \subset \text{Br}(k)$ coming from $\text{Gal}(K(W)/k) \cong G$. Let us consider a decomposition $C \cong \mu_{p^{s_1}} \times \cdots \times \mu_{p^{s_r}}$, and let ε map to $(\varepsilon_i)_i$ via the isomorphism $H^2(G, C) \cong \bigoplus_i H^2(G, \mu_{p^{s_i}})$. Take $\gamma_i = \text{inf}(\varepsilon_i)$.

(i) Let $C = \mu_{p^n}$. From $H' \cap C = \{1\}$ it follows that H' is isomorphic to G' . Therefore, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_{p^n} & \longrightarrow & H & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_{p^n} & \longrightarrow & H/H' & \longrightarrow & G/G' \longrightarrow 1. \end{array}$$

Let $\varepsilon' \in H^2(G/G', \mu_{p^n})$ be the class of the second row of the above diagram. Then we have the homomorphism $\delta : H^2(G/G', \mu_{p^n}) \rightarrow H^2(G, \mu_{p^n})$ such that $\delta(\varepsilon') = \varepsilon$. Note that ε' is not always split, since μ_{p^n} may be contained in a bigger cyclic subgroup of H/H' , and we know that extensions of that kind are not split. Hence ε is not always split and we can not apply Theorem 2.4 at this point.

Thus, we need to take a roundabout approach, involving the theory of obstructions of Galois embedding problems. Let us recall some basics. The embedding problem related to $K(W)/k$ and ε consists in determining whether there exists a Galois algebra (or a Galois extension) L , such that $K(W)$ is contained in L , H is isomorphic to $\text{Gal}(L/k)$, and the homomorphism of restriction of L on $K(W)$ coincides with the epimorphism $H \rightarrow G$. The element $\gamma = \text{inf}(\varepsilon) \in \text{Br}(k)$ is called the obstruction to this embedding problem. It is known that the splitting of the obstruction in the Brauer group is necessary and sufficient condition for the solvability of such types of embedding problems (called Brauer problems, see [Le]).

Now, we have that $\text{inf}(\varepsilon')$ is the obstruction to the embedding problem given by ε' . Let us write the decomposition of H/H' as a direct product of cyclic groups: $H/H' \cong \mu_{p^{m_1}} \times \cdots \times \mu_{p^{m_s}}$. For any m_j , set $\mu_{p^{i_j}} = \mu_{p^{m_j}} \cap \mu_{p^n}$, and consider the embedding problem given by the group extension $1 \rightarrow \mu_{p^{i_j}} \rightarrow \mu_{p^{m_j}} \rightarrow \mu_{p^{m_j-i_j}} \rightarrow 1$ and some $\mu_{p^{m_j-i_j}}$ extension L/k .

Since k contains a primitive $p^{m_j-i_j}$ -th root of unity, we may assume that $L/k = k(\sqrt[p^{m_j-i_j}]{a})/k$ for a certain element $a \in k^* \setminus k^{*p}$. In this way, we obtain an equivalent embedding problem given by $1 \rightarrow \mu_{p^{m_j-1}} \rightarrow \mu_{p^{m_j}} \rightarrow \mu_p \rightarrow 1$ and $k(\sqrt[p]{a})/k$. It is well-known (see [Al, Theorem 11]) that the obstruction to the latter problem is equal to the Hilbert symbol $(a, \zeta_{p^{m_j-1}})_{p^{m_j-1}}$ which is trivial for any a , since $\zeta_{p^{m_j}} = \sqrt[p]{\zeta_{p^{m_j-1}}}$ is in k (recall that $m = \max\{m_j : 1 \leq j \leq s\}$). Therefore, there exists a solution to the embedding problem given by ε' that is isomorphic to the k -compositum of the

solutions to the embedding problems given by $1 \rightarrow \mu_{p^{i_j}} \rightarrow \mu_{p^{m_j}} \rightarrow \mu_{p^{m_j-i_j}} \rightarrow 1$ for $1 \leq j \leq s$. Hence $\inf(\varepsilon') = 0$.

Next, observe that the inflation map $\inf : H^2(G/G', \mu_{p^n}) \rightarrow \text{Br}(k)$ factors through $\delta : H^2(G/G', \mu_{p^n}) \rightarrow H^2(G, \mu_{p^n})$, i.e., we have the following commutative diagram

$$\begin{array}{ccc} H^2(G/G', \mu_{p^n}) & \xrightarrow{\delta} & H^2(G, \mu_{p^n}) \\ \downarrow \inf & & \downarrow \inf \\ \text{Br}(k) & \xrightarrow{i} & \text{Br}(k), \end{array}$$

where i is an injection. Therefore, $\gamma = \inf(\varepsilon) = \inf \delta(\varepsilon') = \inf(\varepsilon') = 0$. From Theorem 2.4 now it follows that $K(H)$ is rational over $K(G)$.

(ii) Let us consider a decomposition $C \cong \mu_{p^{s_1}} \times \cdots \times \mu_{p^{s_r}}$. If $\mu_{p^{s_i}} \cap \mu_{p^{m_i}} \neq \{1\}$ for some cyclic subgroup $\mu_{p^{m_i}}$ of H/H' , then $m_i \leq n + e$, as p^e is the exponent of G/G' . Since K contains a primitive p^{n+e} -th root of unity, we can apply case (i) and get that all the γ_i 's are trivial. Therefore, $K(H)$ is rational over $K(G)$ by Theorem 2.4. \square

Theorem 2.6. (Kang [Ka1, Theorem 4.1]) *Let p be a prime number, m, n and r are positive integers, $k = 1 + p^r$ if $(p, r) \neq (2, 1)$ (resp. $k = -1 + 2^r$ with $r \geq 2$). Let G be a split metacyclic p -group of order p^{m+n} and exponent p^e defined by $G = \langle \sigma, \tau : \sigma^{p^m} = \tau^{p^n} = 1, \tau^{-1}\sigma\tau = \sigma^k \rangle$. Let K be any field such that $\text{char} K \neq p$ and K contains a primitive p^e -th root of unity, and let ζ be a primitive p^m -th root of unity. Then $K(x_0, x_1, \dots, x_{p^n-1})^G$ is rational over K , where G acts on x_0, \dots, x_{p^n-1} by*

$$\begin{aligned} \sigma & : x_i \mapsto \zeta^{k^i} x_i, \\ \tau & : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^n-1} \mapsto x_0. \end{aligned}$$

Theorem 2.7. ([Ha1, Ha2]) *Let K be any field and G a finite subgroup of $\text{GL}(2, \mathbb{Z})$. Then the fixed field $K(x, y)^G$ under monomial action of G is rational over K .*

Finally, we give a Lemma, which can be extracted from some proofs in [Ka2, HuK].

Lemma 2.8. *Let τ be a cyclic group of order $n > 1$, acting on $L(v_1, \dots, v_{n-1})$, the rational function field of $n - 1$ variables over a field L such that*

$$\tau : v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{n-1} \mapsto (v_1 \cdots v_{n-1})^{-1} \mapsto v_1.$$

If L contains a primitive n th root of unity ξ , then $K(v_1, \dots, v_{n-1}) = K(s_1, \dots, s_{n-1})$ where $\tau : s_i \mapsto \xi^i s_i$ for $1 \leq i \leq n - 1$.

Proof. Define $w_0 = 1 + v_1 + v_1v_2 + \cdots + v_1v_2 \cdots v_{n-1}$, $w_1 = (1/w_0) - 1/n$, $w_{i+1} = (v_1v_2 \cdots v_i/w_0) - 1/n$ for $1 \leq i \leq n-1$. Thus $K(v_1, \dots, v_{n-1}) = K(w_1, \dots, w_n)$ with $w_1 + w_2 + \cdots + w_n = 0$ and

$$\tau : w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{n-1} \mapsto w_n \mapsto w_1.$$

Define $s_i = \sum_{1 \leq j \leq n} \xi^{-ij} w_j$ for $1 \leq i \leq n-1$. Then $K(w_1, \dots, w_n) = K(s_1, \dots, s_{n-1})$ and $\tau : s_i \mapsto \xi^i s_i$ for $1 \leq i \leq n-1$. \square

3. THE CYCLIC EXTENSIONS OF THE METACYCLIC p -GROUPS

We are going to consider first the central cyclic extensions of the abelian metacyclic p -groups. Let G be an abelian metacyclic p -group generated by two elements σ and τ with relations $\sigma^{p^a} = \tau^{p^b} = 1$ and $\tau^{-1}\sigma\tau = \sigma$. Let \tilde{G} be a central cyclic extension of G by C_{p^t} such that the pre-image of $[\sigma, \tau] = \sigma^{-1}\tau^{-1}\sigma\tau$ in \tilde{G} is of order p^t . Then we have the group extension

$$(3.1) \quad 1 \longrightarrow C_{p^t} \longrightarrow \tilde{G} \longrightarrow G \cong C_{p^a} \times C_{p^b} \longrightarrow 1,$$

where $C_{p^t} = \langle \rho \rangle \leq Z(\tilde{G})$, $t \leq b \leq a$ and the pre-image of $[\sigma, \tau] = \sigma^{-1}\tau^{-1}\sigma\tau$ in \tilde{G} is of order p^t . If we put $|\tilde{G}| = p^n$, we get $n = a + b + t$. According to [AMM] any 2-generator group of order p^n and nilpotency class 2 is a central extension of the form (3.1). Moreover, for any positive partition (a, b, t) of n , the set of nonisomorphic central extensions of the form (3.1) with nilpotency class exactly 2 is nonempty [AMM, Lemma 2.2].

Any group \tilde{G} then has the presentation

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{ip^\alpha}, \tau^{p^b} = \rho^{jp^\beta}, \rho^{p^t} = 1, [\sigma, \tau] = \rho^l, \rho \text{ central} \rangle,$$

where i, j, l are positive integers, $0 \leq i, j, l < p^t$, $\gcd(ijl, p) = 1$, $0 \leq \alpha, \beta \leq t$; $a \geq b \geq t \geq 1$ and $a + b + t = n$. Note that the commutator subgroup \tilde{G}' is cyclic and is generated by ρ^l .

Bacon and Kappe [BK] give a classification of these groups, but with some omissions that have been corrected recently in [AMM]. We need not the full classification for our purposes, so we will not give it in our paper. Instead, we write the following result from [AMM] which gives a more convenient form of the above presentations.

Proposition 3.1. ([AMM, Proposition 3.1]) *Fix $a \geq b \geq t \geq 1$, let $0 \leq \alpha, \beta \leq t$, and let i, j, l be positive integers, $0 \leq i, j, l < p^t$, with $\gcd(ijl, p) = 1$. Then the groups*

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{ip^\alpha}, \tau^{p^b} = \rho^{jp^\beta}, \rho^{p^t} = 1, [\sigma, \tau] = \rho^l, \rho - \text{central} \rangle,$$

and

$$\tilde{H} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^\alpha}, \tau^{p^b} = \rho^{p^\beta}, \rho^{p^t} = 1, [\sigma, \tau] = \rho, \rho - \text{central} \rangle,$$

are isomorphic.

Now, let G be any nonabelian metacyclic p -group generated by two elements σ and τ with relations $\sigma^{p^a} = 1, \tau^{p^b} = \sigma^{p^c}$ and $\tau^{-1}\sigma\tau = \sigma^k$ for $k = \varepsilon + p^r$ where $\varepsilon = 1$ if p is odd, $\varepsilon = \pm 1$ if $p = 2$, and $a, b, c, r \geq 0$ are subject to some restrictions. For example, we have $a, b, c \geq r$. For the description of these restrictions see e.g. [Ka1, p. 564].

Let t be a positive integer such that $t \leq \min\{a, b\}$ and let \tilde{G} be a central cyclic extension of a nonabelian metacyclic p -group G by C_{p^t} such that the pre-image of $\sigma^{-(k-1)}[\sigma, \tau] = \sigma^{-k}\tau^{-1}\sigma\tau$ in \tilde{G} is of order p^t . Then we have the group extension

$$(3.2) \quad 1 \longrightarrow C_{p^t} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

where $C_{p^t} = \langle \rho \rangle \leq Z(\tilde{G})$.

The group \tilde{G} then has the following presentation:

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{ip^\alpha}, \tau^{p^b} = \sigma^{p^c} \rho^{jp^\beta}, \rho^{p^t} = 1, \tau^{-1}\sigma\tau = \sigma^k \rho^l, \rho - \text{central} \rangle,$$

where i, j, l are positive integers, $0 \leq i, j, l < p^t$, $\gcd(ijl, p) = 1, 0 \leq \alpha, \beta \leq t$ and $k = \varepsilon + p^r$.

Proposition 3.2. *Fix $a \geq t, b \geq t, 0 \leq c \leq a, r \leq \min\{a, b, c\}$. Let $0 \leq \alpha, \beta \leq t$, and let i, j, l be positive integers, $0 \leq i, j, l < p^t$, with $\gcd(ijl, p) = 1$. Then there exist positive integers $m, n : 0 < m < p^a, 0 < n < p^t, \gcd(mn, p) = 1$ such that the group*

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{ip^\alpha}, \tau^{p^b} = \sigma^{p^c} \rho^{jp^\beta}, \rho^{p^t} = 1, \tau^{-1}\sigma\tau = \sigma^k \rho^l, \rho - \text{central} \rangle,$$

is isomorphic to the group

$$(3.3) \quad \tilde{H} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{np^\alpha}, \tau^{p^b} = \sigma^{mp^c} \rho^{p^\beta}, \rho^{p^t} = 1, \tau^{-1}\sigma\tau = \sigma^k \rho, \rho - \text{central} \rangle.$$

Proof. If $\beta = t$, set $m = 1, \sigma_1 = \sigma, \tau_1 = \tau$. Choose an integer l_1 such that $ll_1 \equiv 1 \pmod{p^t}$, i.e., $\rho = (\rho^l)^{l_1}$. Define $n = il_1$ and $\rho_1 = \rho^l$. Then the elements σ_1, τ_1, ρ_1 of \tilde{G} satisfy the same relations as $\sigma, \tau, \rho \in \tilde{H}$. Whence we have an onto homomorphism

$H \rightarrow G$ that maps $\sigma \mapsto \sigma_1, \tau \mapsto \tau_1$ and $\rho \mapsto \rho_1$. Since the two groups have the same order, this map is an isomorphism.

If $\beta < t$, then pick s such that $ls \equiv j \pmod{p^{t-\beta}}$, and set $\sigma_1 = \sigma^s, \tau_1 = \tau$, and $\rho_1 = \rho^{ls}$. Then $\tau_1^{-1}\sigma_1\tau_1 = \sigma_1^k\rho_1, \tau_1^{p^b} = \sigma^{p^c}\rho^{jp^\beta} = \sigma_1^{mp^c}\rho^{lsp^\beta} = \sigma_1^{mp^c}\rho_1^{p^\beta}$ for m such that $ms \equiv 1 \pmod{\text{ord}(\sigma)}$ (i.e., $\sigma = \sigma_1^m$); and $\sigma_1^{p^a} = \rho_1^{np^a}$, where $n = isu$ for some u such that $uls \equiv 1 \pmod{p^t}$ (i.e., $\rho = \rho_1^u$). Again, we obtain a homomorphism from \tilde{H} onto \tilde{G} , showing that $\tilde{G} \cong \tilde{H}$. \square

We need to calculate the commutator subgroup \tilde{G}' in order to apply Theorem 2.5 effectively.

Proposition 3.3. *Let the group \tilde{G} be isomorphic to the group \tilde{H} with a presentation of the form (3.3). The commutator subgroup \tilde{H}' is cyclic and is generated by $\sigma^{k-1}\rho$.*

Proof. First, note that $\tau^{-i}\sigma^j\tau^i = \sigma^{jk^i}\rho^{j \cdot w_i}$ for $1 \leq i \leq \text{ord}(\tau), 1 \leq j \leq \text{ord}(\sigma)$ and $w_i = 1 + k + k^2 + \dots + k^{i-1}$.

Let us calculate now an arbitrary commutator:

$$\begin{aligned} A &= [\tau^i\sigma^j\rho^l, \tau^u\sigma^v\rho^w] = [\tau^i\sigma^j, \tau^u\sigma^v] = (\tau^i\sigma^j)^{-1}(\tau^u\sigma^v)^{-1}(\tau^i\sigma^j)(\tau^u\sigma^v) \\ &= \sigma^{-j}(\tau^{-i}\sigma^{-v}\tau^i)(\tau^{-u}\sigma^j\tau^u)\sigma^v = \sigma^{-j}(\sigma^{-vk^i}\rho^{-vw_i})(\sigma^{jk^u}\rho^{jw_u})\sigma^v \\ &= \sigma^B\rho^C \quad \text{for} \\ B &= -j + jk^u - vk^i + v, \\ C &= -vw_i + jw_u. \end{aligned}$$

We have now $B = j(k^u - 1) - v(k^i - 1) = j(k - 1)w_u - v(k - 1)w_i = (k - 1)C$, so $A = (\sigma^{k-1}\rho)^C$. We are done. \square

In the proof of Theorem 1.5 we are going to consider different cases which appear according to the values of α, β, c and ε . Namely, the group \tilde{G} will be equal to one of the following 16 groups.

- (1) $\varepsilon = 1, c = a, \alpha = \beta = t$
 $G_1 = \langle \sigma, \tau, \rho : \sigma^{p^a} = \tau^{p^b} = \rho^{p^t} = 1, \tau^{-1}\sigma\tau = \sigma^{1+p^r}\rho \rangle,$
- (2) $\varepsilon = 1, c = a, \alpha = t, \beta < t$
 $G_2 = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^t} = 1, \tau^{p^b} = \rho^{p^\beta}, \tau^{-1}\sigma\tau = \sigma^{1+p^r}\rho \rangle,$

- (3) $\varepsilon = 1, c = a, \alpha < t, \beta = t$
 $G_3 = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^a}, \tau^{p^b} = \rho^{p^t} = 1, \tau^{-1}\sigma\tau = \sigma^{1+p^r}\rho \rangle,$
- (4) $\varepsilon = 1, c = a, \alpha < t, \beta < t$
 $G_4 = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^a}, \tau^{p^b} = \rho^{p^\beta}, \rho^{p^t} = 1, \tau^{-1}\sigma\tau = \sigma^{1+p^r}\rho \rangle,$
- (5) $\varepsilon = -1, c = a, \alpha = \beta = t$
 $G_5 = \langle \sigma, \tau, \rho : \sigma^{2^a} = \tau^{2^b} = \rho^{2^t} = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^r}\rho \rangle,$
- (6) $\varepsilon = -1, c = a, \alpha = t, \beta < t$
 $G_6 = \langle \sigma, \tau, \rho : \sigma^{2^a} = \rho^{2^t} = 1, \tau^{2^b} = \rho^{2^\beta}, \tau^{-1}\sigma\tau = \sigma^{-1+2^r}\rho \rangle,$
- (7) $\varepsilon = -1, c = a, \alpha < t, \beta = t$
 $G_7 = \langle \sigma, \tau, \rho : \sigma^{2^a} = \rho^{2^\alpha}, \tau^{2^b} = \rho^{2^t} = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^r}\rho \rangle,$
- (8) $\varepsilon = -1, c = a, \alpha < t, \beta < t$
 $G_8 = \langle \sigma, \tau, \rho : \sigma^{2^a} = \rho^{2^\alpha}, \tau^{2^b} = \rho^{2^\beta}, \rho^{2^t} = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^r}\rho \rangle,$
- (9) $\varepsilon = 1, c < a, \alpha = \beta = t$
 $G_9 = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^t} = 1, \tau^{p^b} = \sigma^{p^c}, \tau^{-1}\sigma\tau = \sigma^{1+p^r}\rho \rangle,$
- (10) $\varepsilon = 1, c < a, \alpha = t, \beta < t$
 $G_{10} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^t} = 1, \tau^{p^b} = \sigma^{p^c}\rho^{p^\beta}, \tau^{-1}\sigma\tau = \sigma^{1+p^r}\rho \rangle,$
- (11) $\varepsilon = 1, c < a, \alpha < t, \beta = t$
 $G_{11} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^\alpha}, \tau^{p^b} = \sigma^{p^c}, \rho^{p^t} = 1, \tau^{-1}\sigma\tau = \sigma^{1+p^r}\rho \rangle,$
- (12) $\varepsilon = 1, c < a, \alpha < t, \beta < t$
 $G_{12} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^\alpha}, \tau^{p^b} = \sigma^{p^c}\rho^{p^\beta}, \rho^{p^t} = 1, \tau^{-1}\sigma\tau = \sigma^{1+p^r}\rho \rangle,$
- (13) $\varepsilon = -1, c < a, \alpha = \beta = t$
 $G_{13} = \langle \sigma, \tau, \rho : \sigma^{2^a} = \rho^{2^t} = 1, \tau^{2^b} = \sigma^{2^c}, \tau^{-1}\sigma\tau = \sigma^{-1+2^r}\rho \rangle,$
- (14) $\varepsilon = -1, c < a, \alpha = t, \beta < t$
 $G_{14} = \langle \sigma, \tau, \rho : \sigma^{2^a} = \rho^{2^t} = 1, \tau^{2^b} = \sigma^{2^c}\rho^{2^\beta}, \tau^{-1}\sigma\tau = \sigma^{-1+2^r}\rho \rangle,$
- (15) $\varepsilon = -1, c < a, \alpha < t, \beta = t$
 $G_{15} = \langle \sigma, \tau, \rho : \sigma^{2^a} = \rho^{2^\alpha}, \tau^{2^b} = \sigma^{2^c}, \rho^{2^t} = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^r}\rho \rangle,$
- (16) $\varepsilon = -1, c < a, \alpha < t, \beta < t$
 $G_{16} = \langle \sigma, \tau, \rho : \sigma^{2^a} = \rho^{2^\alpha}, \tau^{2^b} = \sigma^{2^c}\rho^{2^\beta}, \rho^{2^t} = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^r}\rho \rangle,$

4. PROOF OF THEOREM 1.4

The case (i) follows from Theorem 2.3.

(ii) We will divide the proof into two steps.

Step I. Let $C = C_{p^{\alpha_1}} \times \cdots \times C_{p^{\alpha_s}} \leq Z(\tilde{G})$. Denote by σ and τ the preimages of the generators of G and by ρ_1, \dots, ρ_s the generators of C , i.e., $\rho_i^{p^{\alpha_i}} = 1$. Then $[\sigma, \tau] = \rho_1^{\beta_1} \cdots \rho_s^{\beta_s}$ for $\beta_i \geq 0$, and we can assume for abuse of notation that $p^t = \text{ord}(\rho_1^{\beta_1}) = \max\{\text{ord}(\rho_1^{\beta_1}), \dots, \text{ord}(\rho_s^{\beta_s})\}$. It follows that $\tilde{G}' \cap \langle \rho_2, \dots, \rho_s \rangle = \{1\}$. Now we can apply Theorem 2.5 reducing the rationality problem of $K(\tilde{G})$ over K to the rationality problem of $K(\tilde{G}/\langle \rho_2, \dots, \rho_k \rangle)$ over K . Note that $\exp(\langle \rho_2, \dots, \rho_k \rangle) \leq p^t \leq p^a = \exp(\tilde{G}_1/\tilde{G}'_1)$ for $\tilde{G}_1 = \tilde{G}/\langle \rho_2, \dots, \rho_k \rangle$.

Therefore, we will assume that \tilde{G} has the presentation

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{ip^\alpha}, \tau^{p^b} = \rho^{jp^\beta}, \rho^{p^t} = 1, [\sigma, \tau] = \rho^l, \rho - \text{central} \rangle,$$

where i, j, l are positive integers, $0 \leq i, j, l < p^t$, $\gcd(ijl, p) = 1$, $0 \leq \alpha, \beta \leq t$; $a \geq b \geq t \geq 1$. Note that the commutator subgroup \tilde{G}' is cyclic and is generated by ρ^l .

Moreover, from Proposition 3.1 it follows that we can assume that \tilde{G} is 2-generator p -group of nilpotency class 2 given by the presentation

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^\alpha}, \tau^{p^b} = \rho^{p^\beta}, \rho^{p^t} = 1, [\sigma, \tau] = \rho, \rho - \text{central} \rangle,$$

where $0 \leq \alpha, \beta \leq t$ and $a \geq b \geq t$.

Define $m = a + t - \alpha$ and $n = b + t - \beta$. Then $p^m = \text{ord}(\sigma)$ and $p^n = \text{ord}(\tau)$. According to [AMM] the center of \tilde{G} is

$$Z(\tilde{G}) = \{\sigma^{ip^t} \tau^{jp^t} \rho^k : 1 \leq i \leq p^{m-t}, 1 \leq j \leq p^{n-t}, 1 \leq k \leq p^t\}.$$

Consider the following two possibilities for α .

Case 1. Let $\alpha = t$, i.e., $\sigma^{p^a} = 1$. Then $\tilde{G}' \cap \langle \sigma^{p^t} \rangle = \{1\}$, so from Theorem 2.5 it follows that we can reduce the rationality problem to one related to the group generated by elements σ, τ and ρ such that $\sigma^{p^t} = 1, \tau^{p^b} = \rho^{p^\beta}, \rho^{p^t} = 1, [\sigma, \tau] = \rho, \rho - \text{central}$.

Case 2. Let $\alpha < t$, i.e., $\sigma^{p^a} = \rho^{p^\alpha} \neq 1$. If $a = t$, we obtain that \tilde{G} is generated by elements σ, τ and ρ such that $\sigma^{p^t} = \rho^{p^\alpha}, \tau^{p^b} = \rho^{p^\beta}, \rho^{p^t} = 1, [\sigma, \tau] = \rho, \rho - \text{central}$. Let $a > t$. (Recall that $a \geq b \geq t$.) Then $\sigma^{p^{a-1}} = \sigma^{p^{a-t-1} \cdot p^t} \in Z(\tilde{G})$ and we get $(\sigma^{p^{a-1}} \rho^{-p^{\alpha-1}})^p = 1$. Then $\tilde{G}' \cap \langle \sigma^{p^{a-1}} \rho^{-p^{\alpha-1}} \rangle = \{1\}$, so we can apply Theorem 2.5 reducing the rationality problem of $K(\tilde{G})$ to the rationality problem of $K(\tilde{G}_1)$ over K , where $\tilde{G}_1 \cong \tilde{G}/\langle \sigma^{p^{a-1}} \rho^{-p^{\alpha-1}} \rangle$. The group \tilde{G}_1 is generated by elements σ, τ and ρ such that $\sigma^{p^{a-1}} = \rho^{p^{\alpha-1}}, \tau^{p^b} = \rho^{p^\beta}, \rho^{p^t} = 1, [\sigma, \tau] = \rho, \rho - \text{central}$. We can proceed in the

same way until we obtain either a group \tilde{G}_k such that $\sigma^{p^t} = \rho^{p^{\alpha-u}}$ for $u = a - t$ and $\alpha \geq u$; or a metacyclic p -group such that $\sigma^{p^{a-l}} = \rho$ for some l . For the second group we can apply Theorem 1.2.

Similarly, we can consider the cases $\beta = t$ and $\beta < t$. In this way, we may assume that \tilde{G} is generated by elements σ, τ and ρ such that $\sigma^{p^t} = \rho^{p^\alpha}, \tau^{p^t} = \rho^{p^\beta}, \rho^{p^t} = 1, [\sigma, \tau] = \rho, \rho$ – central. Note that anytime we applied Theorem 2.5 so far, the condition for the root of unity is satisfied, since $\exp(\tilde{G}/\tilde{G}') \leq p^a$.

Step II. We are going to consider the regular representation of \tilde{G} . Let V be a K -vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in \tilde{G}} K \cdot x(g)$ where \tilde{G} acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in \tilde{G}$. Thus $K(V)^{\tilde{G}} = K(x(g) : g \in \tilde{G})^{\tilde{G}} = K(\tilde{G})$.

For $m = 2t - \alpha$ and $n = 2t - \beta$ define $\zeta = \zeta_{p^m}$, a primitive p^m -th root of unity; $\eta = \zeta^{p^{m-n}}$, a primitive p^n -th root of unity; and $\xi = \zeta^{p^{t-\alpha}}$, a primitive p^t -th root of unity. Since K contains a primitive p^{2t} -th root of unity, we have $\zeta, \eta, \xi \in K$. For $0 \leq i \leq p^t - 1$ define $x_i \in V^*$ by

$$x_i = \sum_{j,k} \eta^{-j-kp^{b-\beta}} x(\sigma^i \tau^j \rho^k),$$

where $0 \leq j \leq p^t - 1, 0 \leq k \leq p^t - 1$. The actions of σ, τ and ρ are given by

$$\sigma : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^t-1} \mapsto \xi^{p^\alpha} x_0,$$

$$\tau : x_i \mapsto \eta^{\xi^i} x_i,$$

$$\rho : x_i \mapsto \xi x_i,$$

for $0 \leq i \leq p^t - 1$. We find that $Y = \bigoplus_{0 \leq i \leq p^t-1} K \cdot x_i$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ is rational over K .

Define

$$y_0 = x_0^{p^t}, y_1 = x_1/x_0, y_2 = x_2/x_1, \dots, y_{p^t-1} = x_{p^t-1}/x_{p^t-2}.$$

We have now $K(x_0, \dots, x_{p^t-1})^{(\rho)} = K(y_0, \dots, y_{p^t-1})$ and

$$\sigma : y_0 \mapsto y_1^{p^t} y_0, y_1 \mapsto y_2 \mapsto \cdots \mapsto y_{p^t-1} \mapsto \xi^{p^\alpha} / (y_1 \cdots y_{p^t-1}) \mapsto y_1,$$

$$\tau : y_0 \mapsto \eta^{p^t} y_0, y_i \mapsto \xi y_i,$$

for $1 \leq i \leq p^t - 1$. From Theorem 2.2 follows that if $K(y_1, \dots, y_{p^t-1})^{(\sigma, \tau)}$ is rational over K , so is $K(y_0, \dots, y_{p^t-1})^{(\sigma, \tau)}$.

Define

$$z_2 = y_2/y_1, z_3 = y_3/y_2, \dots, z_{p^t-1} = y_{p^t-1}/y_{p^t-2}.$$

We have

$$\sigma : y_1 \mapsto z_2 y_1, z_2 \mapsto z_3 \mapsto \dots \mapsto z_{p^t-1} \mapsto \xi^{p^\alpha} / (y_1 y_2 \dots y_{p^t-1}^2) = \xi^{p^\alpha} / (y_1^{p^t} z_2^{p^t-1} \dots z_{p^t-1}^2),$$

$$\tau : y_1 \mapsto \xi y_1, z_i \mapsto z_i,$$

for $2 \leq i \leq p^t - 1$.

Define $z_1 = y_1^{p^t} \xi^{-p^\alpha}$, i.e., $y_1^{p^t} = z_1 \xi^{p^\alpha}$. We have now $K(y_1, \dots, y_{p^t-1})^{\langle \tau \rangle} = K(z_1, \dots, z_{p^t-1})$ and

$$\sigma : z_1 \mapsto z_2^{p^t} z_1, z_2 \mapsto z_3 \mapsto \dots \mapsto z_{p^t-1} \mapsto 1 / (z_1 z_2^{p^t-1} z_3^{p^t-2} \dots z_{p^t-1}^2).$$

Define $s_1 = z_2, s_i = \tau^{i-1} \cdot z_2$ for $2 \leq i \leq p^t - 1$. Then $K(z_i : 1 \leq i \leq p^t - 1) = K(s_i : 1 \leq i \leq p^t - 1)$ and

$$\tau : s_1 \mapsto s_2 \mapsto \dots \mapsto s_{p^t-1} \mapsto (s_1 s_2 \dots s_{p^t-1})^{-1}.$$

The action of τ can be linearized according to Lemma 2.8. Thus $K(s_i : 1 \leq i \leq p^t - 1)^{\langle \tau \rangle}$ is rational over K by Theorem 1.1. We are done.

5. PROOF OF THEOREM 1.5

The case (i) follows from Theorem 2.3.

(ii) We will divide the proof into two steps.

Step I. Assume that \tilde{G} has the following presentation:

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^\alpha}, \tau^{p^b} = \sigma^{p^c} \rho^{p^\beta}, \rho^{p^t} = 1, \tau^{-1} \sigma \tau = \sigma^k \rho, \rho - \text{central} \rangle,$$

where $a \geq t, b \geq t, 0 \leq \alpha, \beta \leq t$ and $k = \varepsilon + p^r$. Note that we have the relations $\tau^{-i} \sigma \tau^i = \sigma^{k^i} \rho^{w_i}$, where $w_i = 1 + k + k^2 + \dots + k^{i-1}$ and $1 \leq i \leq \text{ord}(\tau)$. Clearly, \tilde{G} is isomorphic to some group \tilde{G}_i for $1 \leq i \leq 16$ given in Section 3. We are going to consider each case separately.

Case 1. $\tilde{G} = G_1$, where G_1 is the group in Section 3. From Proposition 3.3 it follows that \tilde{G}' is cyclic and is generated by $\sigma^{p^r} \rho$.

Subcase 1.a. Let $a \geq r + t$. Then $(\sigma^{p^r} \rho)^{p^{a-r}} = \rho^{p^{a-r}} = 1$, so $\tilde{G}' \cap \langle \rho \rangle = 1$. Theorem 2.5 then implies that we can reduce the rationality problem of $K(\tilde{G})$ to $K(\tilde{G}/\langle \rho \rangle)$ over K , where $\tilde{G}/\langle \rho \rangle$ clearly is a metacyclic p -group.

Subcase 1.b. Let $a < r + t$. Let V be a K -vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in \tilde{G}} K \cdot x(g)$ where \tilde{G} acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in \tilde{G}$. Thus $K(V)^{\tilde{G}} = K(x(g) : g \in \tilde{G})^{\tilde{G}} = K(\tilde{G})$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_{i=0}^{p^a-1} x(\sigma^i), \quad X_2 = \sum_{i=0}^{p^t-1} x(\rho^i).$$

Note that $\sigma \cdot X_1 = X_1$ and $\rho \cdot X_2 = X_2$.

Let $\zeta = \zeta_{p^a} \in K$ be a primitive p^a -th root of unity and define $\xi = \zeta^{p^{a-t}}$. Thus ξ is a primitive p^t -th root of unity. Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^t-1} \xi^{-i} \rho^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p^a-1} \zeta^{-i} \sigma^i \cdot X_2.$$

It follows that

$$\begin{aligned} \sigma & : Y_1 \mapsto Y_1, Y_2 \mapsto \zeta Y_2, \\ \rho & : Y_1 \mapsto \xi Y_1, Y_2 \mapsto Y_2. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup $\langle \sigma, \rho \rangle$.

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2$ for $0 \leq i \leq p^b - 1$. We have now

$$\begin{aligned} \sigma & : x_i \mapsto \xi^{w_i} x_i, \quad y_i \mapsto \zeta^{k_i} y_i \\ \tau & : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^b-1} \mapsto x_0, \\ & \quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^b-1} \mapsto y_0, \\ \rho & : x_i \mapsto \xi x_i, \quad y_i \mapsto y_i, \end{aligned}$$

for $0 \leq i \leq p^b - 1$.

We find that $Y = (\bigoplus_{0 \leq i \leq p^b-1} K \cdot x_i) \oplus (\bigoplus_{0 \leq i \leq p^b-1} K \cdot y_i)$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i : 0 \leq i \leq p^b - 1)^{\tilde{G}}$ is rational over K .

For $1 \leq i \leq p^b - 1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \leq i \leq p^b - 1) = K(x_0, y_0, u_i, v_i : 1 \leq i \leq p^b - 1)$ and for every $g \in \tilde{G}$

$$g \cdot x_0 \in K(u_i, v_i : 1 \leq i \leq p^t - 1) \cdot x_0, \quad g \cdot y_0 \in K(u_i, v_i : 1 \leq i \leq p^b - 1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \leq i \leq p^b - 1)$ is invariant by the action of \tilde{G} . Thus $K(x_i, y_i : 0 \leq i \leq p^b - 1)^{\tilde{G}} = K(u_i, v_i : 1 \leq i \leq p^b - 1)^{\tilde{G}}(u, v)$ for some u, v such that

$\sigma(v) = \tau(v) = \rho(v) = v$ and $\sigma(u) = \tau(u) = \rho(u) = u$. We have now

$$\begin{aligned} \sigma & : u_i \mapsto \xi^{k^{i-1}} u_i, \quad v_i \mapsto \zeta^{k^i - k^{i-1}} v_i, \quad v \mapsto v \\ \tau & : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^b-1} \mapsto (u_1 u_2 \cdots u_{p^b-1})^{-1}, \\ & \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^b-1} \mapsto (v_1 v_2 \cdots v_{p^b-1})^{-1}, \\ & \quad v \mapsto v, \\ \rho & : u_i \mapsto u_i, \quad v_i \mapsto v_i, \quad v \mapsto v \end{aligned}$$

for $1 \leq i \leq p^b - 1$. From Theorem 2.2 it follows that if $K(u_i, v_i : 1 \leq i \leq p^b - 1)^{\tilde{G}}(v)$ is rational over K , so is $K(x_i, y_i : 0 \leq i \leq p^b - 1)^{\tilde{G}}$ over K .

Since ρ acts trivially on $K(u_i, v_i : 1 \leq i \leq p^t - 1)$, we find that $K(u_i, v_i : 1 \leq i \leq p^t - 1)^{\tilde{G}} = K(u_i, v_i : 1 \leq i \leq p^t - 1)^{\langle \sigma, \tau \rangle}$.

Recall that $a < r + t$. So we can write $r = a - t + r_1$ for some $r_1 \geq 1$. It follows that $\zeta^{k^{i-1}(k-1)} = \zeta^{k^{i-1}p^{a-t+r_1}} = \xi^{k^{i-1}p^{r_1}}$ for $1 \leq i \leq p^b - 1$.

Define $w_i = v_i / (u_i)^{p^{r_1}}$. We have now that $K(u_i, v_i : 1 \leq i \leq p^b - 1) = K(u_i, w_i : 1 \leq i \leq p^b - 1)$ and

$$\begin{aligned} \sigma & : u_i \mapsto \xi^{k^{i-1}} u_i, \quad w_i \mapsto w_i, \quad v \mapsto v \\ (5.1) \quad \tau & : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^b-1} \mapsto (u_1 u_2 \cdots u_{p^b-1})^{-1}, \\ & \quad w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{p^b-1} \mapsto (w_1 w_2 \cdots w_{p^b-1})^{-1}, \\ & \quad v \mapsto v. \end{aligned}$$

According to Lemma 2.8, we can linearize the action of τ on w_1, \dots, w_{p^b-1} .

Write $L = K(v, u_i : 1 \leq i \leq p^b - 1)$ and consider $L(w_i : 1 \leq i \leq p^b - 1)^{\langle \sigma, \tau \rangle}$. Note that the group $\langle \sigma, \tau \rangle$ acts on the field $L(w_i)$ as $\langle \sigma, \tau \rangle / \langle \sigma^{p^t} \rangle$ and is faithful on L . Thus we may apply Theorem 2.1 to $L(w_i : 1 \leq i \leq p^b - 1)^{\langle \sigma, \tau \rangle}$. It remains to show that $L^{\langle \sigma, \tau \rangle}$ is rational over K .

Let η be a primitive p^{r+t} -th root of unity such that $\xi = \eta^{p^r}$. Whence $\xi^{k^{i-1}} = \eta^{k^i - k^{i-1}}$.

Consider the metacyclic p -group $\tilde{G}_1 = \langle \sigma, \tau : \sigma^{p^{r+t}} = \tau^{p^b} = 1, \tau^{-1} \sigma \tau = \sigma^k \rangle$.

Define $X = \sum_{0 \leq j \leq p^{r+t-1}} \eta^{-j} x(\sigma^j)$, $V_i = \tau^i X$ for $0 \leq i \leq p^t - 1$. It follows that

$$\begin{aligned} \sigma & : V_i \mapsto \eta^{k^i} V_i, \\ \tau & : V_0 \mapsto V_1 \mapsto \cdots \mapsto V_{p^b-1} \mapsto V_0. \end{aligned}$$

Note that $K(V_0, V_1, \dots, V_{p^b-1})^{\tilde{G}_1}$ is rational by Theorem 2.6.

Define $v_i = V_i/V_{i-1}$ for $1 \leq i \leq p^b - 1$. Then $K(V_0, V_1, \dots, V_{p^b-1})^{\tilde{G}_1} = K(v_1, v_2, \dots, v_{p^b-1})^{\tilde{G}_1}(V)$ where

$$\begin{aligned}\sigma &: v_i \mapsto \eta^{k^i - k^{i-1}} v_i, \quad V \mapsto V \\ \tau &: v_1 \mapsto v_2 \mapsto \dots \mapsto v_{p^b-1} \mapsto (v_1 v_2 \dots v_{p^b-1})^{-1}, \quad V \mapsto V.\end{aligned}$$

Whence $K(u_1, \dots, u_{p^b-1})^{\tilde{G}}(v) \cong K(v_1, \dots, v_{p^b-1})^{\tilde{G}_1}(V) = K(V_0, V_1, \dots, V_{p^b-1})^{\tilde{G}_1}$ is rational over K .

Case 2. $\tilde{G} = G_2$. The proof is almost the same as in Case 1. Here only the action of τ on x_i 's and u_i 's is changed:

$$\tau : x_0 \mapsto x_1 \mapsto \dots \mapsto x_{p^b-1} \mapsto \xi^{p^\beta} x_0.$$

and, respectively,

$$\tau : u_1 \mapsto u_2 \mapsto \dots \mapsto u_{p^b-1} \mapsto \xi^{p^\beta} (u_1 u_2 \dots u_{p^b-1})^{-1}.$$

Let $\omega \in K$ be a primitive $p^{b+t-\beta}$ -th root of unity such that $\omega^{p^{b-\beta}} = \xi$. Then $\xi^{p^\beta} = \omega^{p^b}$. Define $U_i = u_i/\omega$ for $1 \leq i \leq p^b - 1$. Then we have

$$\tau : U_1 \mapsto U_2 \mapsto \dots \mapsto U_{p^b-1} \mapsto (U_1 U_2 \dots U_{p^b-1})^{-1}.$$

Apply the proof of Case 1.

Case 3. $\tilde{G} = G_3$. The subgroup $H = \langle \sigma, \rho \rangle$ is abelian and has an order p^{a+t} . Put $\rho_1 = \sigma^{p^{a-\alpha}} \rho^{-1}$. Then $H \cong \langle \sigma \rangle \times \langle \rho_1 \rangle$, where $\sigma^{p^{a+t-\alpha}} = \rho_1^{p^\alpha} = 1$. We have also $\rho = \sigma^{p^{a-\alpha}} \rho_1^{-1}$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_i x(\rho_1^i), \quad X_2 = \sum_i x(\sigma^i).$$

Note that $\sigma \cdot X_2 = X_2, \rho_1 \cdot X_1 = X_1$.

Let $\zeta_1 \in K$ be a primitive $p^{a+t-\alpha}$ -th root of unity. Put $\xi = \zeta_1^{p^{a-\alpha}}$, a primitive p^t -th root of unity. Let ζ_2 be any primitive p^α -th root of unity. (Note that we will specify ζ_2 a bit later.)

Define $Y_1, Y_2, Y_3 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^{a+t-\alpha}-1} \zeta_1^{-i} \sigma^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p^\alpha-1} \zeta_2^{-i} \rho_1^i \cdot X_3.$$

It follows that

$$\begin{aligned}\sigma & : Y_1 \mapsto \zeta_1 Y_1, \ Y_2 \mapsto Y_2, \\ \rho_1 & : Y_1 \mapsto Y_1, \ Y_2 \mapsto \zeta_2 Y_2, \\ \rho & : Y_1 \mapsto \xi Y_1, \ Y_2 \mapsto \zeta_2^{-1} Y_2.\end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H .

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2$ for $0 \leq i \leq p^b - 1$. We have now

$$\begin{aligned}\sigma & : x_i \mapsto \zeta_1^{k^i} \xi^{w_i} x_i, \ y_i \mapsto \zeta_2^{-w_i} y_i, \\ \tau & : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^b-1} \mapsto x_0, \\ & \quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^b-1} \mapsto y_0, \\ \rho & : x_i \mapsto \xi x_i, \ y_i \mapsto \zeta_2^{-1} y_i.\end{aligned}$$

for $0 \leq i \leq p^b - 1$. We find that $Y = (\bigoplus_{0 \leq i \leq p^b-1} K \cdot x_i) \oplus (\bigoplus_{0 \leq i \leq p^b-1} K \cdot y_i)$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i : 0 \leq i \leq p^b - 1)^{\tilde{G}}$ is rational over K .

For $1 \leq i \leq p^b - 1$, define $U_i = x_i/x_{i-1}$ and $V_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \leq i \leq p^b - 1) = K(x_0, y_0, U_i, V_i : 1 \leq i \leq p^b - 1)$ and for every $g \in \tilde{G}$

$$g \cdot x_0 \in K(U_i, V_i) \cdot x_0, \ g \cdot y_0 \in K(U_i, V_i) \cdot y_0,$$

while the subfield $K(U_i, V_i : 1 \leq i \leq p^b - 1)$ is invariant by the action of \tilde{G} , i.e.,

$$\begin{aligned}\sigma & : U_i \mapsto \zeta_1^{k^i - k^{i-1}} \xi^{k^{i-1}} U_i, \ V_i \mapsto \zeta_2^{-k^{i-1}} V_i, \\ \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^b-1} \mapsto (U_1 \cdots U_{p^b-1})^{-1}, \\ & \quad V_1 \mapsto V_2 \mapsto \cdots \mapsto V_{p^b-1} \mapsto (V_1 \cdots V_{p^b-1})^{-1}, \\ \rho & : U_i \mapsto U_i, \ V_i \mapsto V_i.\end{aligned}$$

for $1 \leq i \leq p^b - 1$. From Theorem 2.2 it follows that if $K(U_i, V_i : 1 \leq i \leq p^b - 1)^{\tilde{G}}$ is rational over K , so is $K(x_i, y_i : 0 \leq i \leq p^b - 1)^{\tilde{G}}$ over K .

Since ρ acts trivially on $K(U_i, V_i)$, we find that $K(U_i, V_i)^{\tilde{G}} = K(U_i, V_i)^{\langle \sigma, \tau \rangle}$.

Subcase 3.a. Let $a - \alpha \leq r$. Thus we can write $r = a - \alpha + r_1$ for some $r_1 \geq 0$. Define $\zeta_2 = \xi^{(1+p^{r_1})p^{t-\alpha}}$, a primitive p^α -th root of unity. Therefore, $\zeta_1^{k-1} = \zeta_1^{p^{a-\alpha+r_1}} = \xi^{p^{r_1}}$ and also $\zeta_1^{k^{i-1}(k-1)} = \xi^{k^{i-1}p^{r_1}}$ for all i .

Define $v_i = U_i^{p^{t-\alpha}} V_i$. Since $\zeta_2 = \xi^{(1+p^{r_1})p^{t-\alpha}}$, we have

$$(5.2) \quad \begin{aligned} \sigma & : U_i \mapsto \xi^{k^{i-1}(1+p^{r_1})} U_i, \quad v_i \mapsto v_i, \\ \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^b-1} \mapsto (U_1 \cdots U_{p^b-1})^{-1}, \\ & \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^b-1} \mapsto (v_1 \cdots v_{p^b-1})^{-1} \end{aligned}$$

for $1 \leq i \leq p^b - 1$. Compare Formula (5.2) with Formula (5.1). They look almost the same. Apply the proof of Case 1.

Subcase 3.b. Let $a - \alpha > r$. Let η be a primitive p^{r+t} -th root of unity such that $\xi = \eta^{p^r}$. Whence $\xi^{k^{i-1}} = \eta^{k^i - k^{i-1}}$. Since $a + t - \alpha > r + t$, we get that $\zeta_1 \eta$ is a primitive $p^{a+t-\alpha}$ -th root of unity. Put $\mu = (\zeta_1 \eta)^{p^{a+t-\alpha-(\alpha+r)}}$, a primitive $p^{\alpha+r}$ -th root of unity, where $\alpha + r < a \leq a + t - \alpha$. Now, put $\zeta_2 = \mu^{p^r}$, a primitive p^α -th root of unity. Thus $\zeta_2^{k^{i-1}} = \mu^{k^i - k^{i-1}}$ for any i .

Define $v_i = U_i^{p^{a+t-\alpha-(\alpha+r)}} V_i$ for $1 \leq i \leq p^b - 1$. It follows that

$$(5.3) \quad \begin{aligned} \sigma & : U_i \mapsto (\zeta_1 \eta)^{k^i - k^{i-1}} U_i, \quad v_i \mapsto v_i, \\ \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^b-1} \mapsto (U_1 \cdots U_{p^b-1})^{-1}, \\ & \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^b-1} \mapsto (v_1 \cdots v_{p^b-1})^{-1} \end{aligned}$$

for $1 \leq i \leq p^b - 1$. Compare Formula (5.3) with Formula (5.1). Considering that $\xi^{k^{i-1}} = \eta^{k^i - k^{i-1}}$, they look almost the same. Apply the proof of Case 1.

Case 4. $\tilde{G} = G_4$. The proof is almost the same as in Case 3. Here only the action of τ is changed. As we already saw in Case 2, this issue is solved with a proper adjustment involving certain root of unity which is in K .

Case 5. $\tilde{G} = G_5$. From Proposition 3.3 it follows that \tilde{G} is cyclic and is generated by $\sigma^{-2+2^r} \rho$. If $a \geq t + 1$ then $(\sigma^{-2+2^r} \rho)^{2^{a-1}} = 1$ and whence $\tilde{G} \cap \langle \rho \rangle = \{1\}$. Theorem 2.5 then implies that we can reduce the rationality problem of $K(\tilde{G})$ to $K(\tilde{G}/\langle \rho \rangle)$ over K , where $\tilde{G}/\langle \rho \rangle$ clearly is a metacyclic p -group. Therefore, we may assume that $a = t$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_{i=0}^{2^t-1} x(\sigma^i), \quad X_2 = \sum_{i=0}^{2^t-1} x(\rho^i).$$

Note that $\sigma \cdot X_1 = X_1$ and $\rho \cdot X_2 = X_2$.

Let ξ be a primitive 2^t -th root of unity. Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{2^t-1} \xi^{-i} \rho^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{2^t-1} \xi^{-i} \sigma^i \cdot X_2.$$

It follows that

$$\begin{aligned} \sigma & : Y_1 \mapsto Y_1, Y_2 \mapsto \xi Y_2, \\ \rho & : Y_1 \mapsto \xi Y_1, Y_2 \mapsto Y_2. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup $\langle \sigma, \rho \rangle$.

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2$ for $0 \leq i \leq 2^b - 1$. We have now

$$\begin{aligned} \sigma & : x_i \mapsto \xi^{w_i} x_i, \quad y_i \mapsto \xi^{k^i} y_i \\ \tau & : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{2^b-1} \mapsto x_0, \\ & \quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{2^b-1} \mapsto y_0, \\ \rho & : x_i \mapsto \xi x_i, \quad y_i \mapsto y_i, \end{aligned}$$

for $0 \leq i \leq 2^b - 1$.

We find that $Y = (\bigoplus_{0 \leq i \leq 2^b-1} K \cdot x_i) \oplus (\bigoplus_{0 \leq i \leq 2^b-1} K \cdot y_i)$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i : 0 \leq i \leq 2^b - 1)^{\tilde{G}}$ is rational over K .

For $1 \leq i \leq 2^b - 1$, define $U_i = x_i/x_{i-1}$ and $V_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \leq i \leq 2^b - 1) = K(x_0, y_0, U_i, V_i : 1 \leq i \leq 2^b - 1)$ and for every $g \in \tilde{G}$

$$g \cdot x_0 \in K(U_i, V_i : 1 \leq i \leq 2^b - 1) \cdot x_0, \quad g \cdot y_0 \in K(U_i, V_i : 1 \leq i \leq 2^b - 1) \cdot y_0,$$

while the subfield $K(U_i, V_i : 1 \leq i \leq 2^b - 1)$ is invariant by the action of \tilde{G} . Thus $K(x_i, y_i : 0 \leq i \leq 2^b - 1)^{\tilde{G}} = K(U_i, V_i : 1 \leq i \leq 2^b - 1)^{\tilde{G}}(u, v)$ for some u, v such that $\sigma(v) = \tau(v) = \rho(v) = v$ and $\sigma(u) = \tau(u) = \rho(u) = u$. We have now

$$\begin{aligned} \sigma & : U_i \mapsto \xi^{k^{i-1}} U_i, \quad V_i \mapsto \xi^{k^i - k^{i-1}} V_i, \quad v \mapsto v \\ \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{2^b-1} \mapsto (U_1 U_2 \cdots U_{2^b-1})^{-1}, \\ & \quad V_1 \mapsto V_2 \mapsto \cdots \mapsto V_{2^b-1} \mapsto (V_1 V_2 \cdots V_{2^b-1})^{-1}, \\ & \quad v \mapsto v, \\ \rho & : U_i \mapsto U_i, \quad V_i \mapsto V_i, \quad v \mapsto v \end{aligned}$$

for $1 \leq i \leq 2^b - 1$. From Theorem 2.2 it follows that if $K(U_i, V_i : 1 \leq i \leq 2^b - 1)^{\tilde{G}}(v)$ is rational over K , so is $K(x_i, y_i : 0 \leq i \leq 2^b - 1)^{\tilde{G}}$ over K .

Since ρ acts trivially on $K(U_i, V_i : 1 \leq i \leq 2^b - 1)$, we find that $K(U_i, V_i : 1 \leq i \leq 2^b - 1)^{\tilde{G}} = K(U_i, V_i : 1 \leq i \leq 2^b - 1)^{\langle \sigma, \tau \rangle}$.

For $1 \leq i \leq 2^b - 1$, define $v_i = V_i/U_i^{k-1}$. Let $\eta \in K$ be a primitive 2^{t+1} -th root of unity such that $\xi = \eta^{-2+2^r}$. Then $\xi^{k^{i-1}} = \eta^{k^i - k^{i-1}}$ for any i . We have now the actions

$$(5.4) \quad \begin{aligned} \sigma & : U_i \mapsto \eta^{k^i - k^{i-1}} U_i, \quad v_i \mapsto v_i, \quad v \mapsto v \\ \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{2^b-1} \mapsto (U_1 U_2 \cdots U_{2^b-1})^{-1}, \\ & \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{2^b-1} \mapsto (v_1 v_2 \cdots v_{2^b-1})^{-1}, \\ & \quad v \mapsto v, \end{aligned}$$

for $1 \leq i \leq 2^b - 1$. Compare Formula (5.4) with Formula (5.3). They look almost the same. Apply the proof of Case 3.

Case 6. $\tilde{G} = G_6$. Similarly to Case 5, we may assume that $a = t$. The proof is almost the same as in Case 5. Here only the action of τ is changed. As we already saw in Case 2, this issue is solved with a proper adjustment involving certain root of unity which is in K .

Case 7. $\tilde{G} = G_7$. From the relation $\tau^{-1} \sigma^{2^a} \tau = \sigma^{-2^a+2^{a+r}} = \sigma^{2^a}$ it follows that the order of σ is 2^{a+1} . Therefore, $\alpha = t - 1$. Note that the subgroup $H = \langle \sigma, \rho \rangle$ is abelian and has an order p^{a+t} . Put $\rho_1 = \sigma^{2^{a+1-t}} \rho^{-1}$. Then $H \cong \langle \sigma \rangle \times \langle \rho_1 \rangle$, where $\sigma^{2^{a+1}} = \rho_1^{2^{t-1}} = 1$. We have also $\rho = \sigma^{2^{a+1-t}} \rho_1^{-1}$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_{i=0}^{2^{a+1}-1} x(\sigma^i), \quad X_2 = \sum_{i=0}^{2^{t-1}-1} x(\rho_1^i).$$

Note that $\sigma \cdot X_1 = X_1$ and $\rho_1 \cdot X_2 = X_2$.

Let ζ_1 be a primitive 2^{a+1} -th root of unity; $\xi = \zeta_1^{2^{a+1-t}}$, a primitive 2^t -th root of unity; and $\mu = \zeta_1^{2^{a+1-t+1}}$, a primitive 2^{t-1} -th root of unity. Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{2^{t-1}-1} \mu^{-i} \rho_1^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{2^{a+1}-1} \zeta_1^{-i} \sigma^i \cdot X_2.$$

It follows that

$$\sigma : Y_1 \mapsto Y_1, Y_2 \mapsto \zeta_1 Y_2,$$

$$\rho : Y_1 \mapsto \mu^{-1}Y_1, Y_2 \mapsto \xi Y_2.$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H .

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2$ for $0 \leq i \leq 2^b - 1$. We have now

$$\begin{aligned} \sigma & : x_i \mapsto \mu^{-w_i}x_i, y_i \mapsto \xi^{w_i}\zeta_1^{k^i}y_i \\ \tau & : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{2^b-1} \mapsto x_0, \\ & y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{2^b-1} \mapsto y_0, \\ \rho & : x_i \mapsto \mu^{-1}x_i, y_i \mapsto \xi y_i, \end{aligned}$$

for $0 \leq i \leq 2^b - 1$. Computations show that for $a = t$ we have $\sigma^{2^{t-1}}(y_i) = \rho^{2^{t-2}}(y_i)$ for any i , which means that $Y = \bigoplus_{0 \leq i \leq 2^b-1} K \cdot y_i$ is not a faithful \tilde{G} -subspace of V^* .

Define $z_i = x_i y_i$ for $1 \leq i \leq 2^b - 1$. It follows that

$$\begin{aligned} \sigma & : x_i \mapsto \mu^{-w_i}x_i, z_i \mapsto (\mu^{-1}\xi)^{w_i}\zeta_1^{k^i}z_i \\ \tau & : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{2^b-1} \mapsto x_0, \\ & z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{2^b-1} \mapsto z_0, \\ \rho & : x_i \mapsto \mu^{-1}x_i, z_i \mapsto \mu^{-1}\xi z_i, \end{aligned}$$

for $0 \leq i \leq 2^b - 1$. Now for $a = t$ we have $\sigma^{2^{t-1}}(z_i) = -\rho^{2^{t-2}}(z_i)$, and for any other value of $a > t$ we have $\sigma^{2^{a-1}}(z_i) \neq \rho^{2^{a-2}}(z_i)$. It follows that $Y = \bigoplus_{0 \leq i \leq 2^b-1} K \cdot z_i$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(z_i : 0 \leq i \leq 2^b - 1)^{\tilde{G}}$ is rational over K .

For $1 \leq i \leq 2^b - 1$ define $U_i = z_i/z_{i-1}$. We have now

$$\begin{aligned} \sigma & : U_i \mapsto (\mu^{-1}\xi)^{k^{i-1}}\zeta_1^{k^i-k^{i-1}}U_i, \\ (5.5) \quad \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{2^b-1} \mapsto (U_1 U_2 \cdots U_{2^b-1})^{-1}, \\ \rho & : U_i \mapsto U_i, \end{aligned}$$

for $1 \leq i \leq 2^b - 1$.

Let η be a primitive 2^{t+1} -th root of unity such that $\mu^{-1}\xi = \eta^{k-1}$. Whence $(\mu^{-1}\xi)^{k^{i-1}} = \eta^{k^i-k^{i-1}}$.

Compare Formula (5.5) with Formula (5.3). They look almost the same. Apply the proof of Case 3.

Cases 8-16. $\tilde{G} = G_i$ for $8 \leq i \leq 16$. It is easily seen that we need to make only minor changes, as we did in Cases 2 and 8, in order to ensure the proper action of τ .

Step II. Consider the general presentation of \tilde{G} . According to Proposition 3.2, we may assume that \tilde{G} has the following presentation:

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{sp^\alpha}, \tau^{p^b} = \sigma^{mp^c} \rho^{p^\beta}, \rho^{p^t} = 1, \tau^{-1} \sigma \tau = \sigma^k \rho, \rho - \text{central} \rangle,$$

where s, m are positive integers, $1 \leq s < p^t, 0 \leq m < p^a, \gcd(sm, p) = 1, 0 \leq \alpha, \beta \leq t$ and $k = \varepsilon + p^r$.

Here we have again 16 cases, which correspond to those in Step I. Since they all can be treated in an unified way, we will consider only Case 3.

Let $c = \beta = t$, i.e., $\tau^{p^b} = 1$. The subgroup $H = \langle \sigma, \rho \rangle$ is abelian and has an order p^{a+t} . Put $\rho_1 = \sigma^{p^{a-\alpha}} \rho^{-s}$. Then $H \cong \langle \sigma \rangle \times \langle \rho_1 \rangle$, where $\sigma^{p^{a+t-\alpha}} = \rho_1^{p^\alpha} = 1$. Let n be an integer such that $ns \equiv 1 \pmod{p^t}$. We have now $\rho = (\rho^s)^n = \sigma^{np^{a-\alpha}} \rho_1^{-n}$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_i x(\rho_1^i), \quad X_2 = \sum_i x(\sigma^i).$$

Note that $\sigma \cdot X_2 = X_2, \rho_1 \cdot X_1 = X_1$.

Let $\zeta_1 \in K$ be a primitive $p^{a+t-\alpha}$ -th root of unity. Put $\xi = \zeta_1^{p^{a-\alpha}}$, a primitive p^t -th root of unity. Let ζ_2 be any primitive p^α -th root of unity. (We will specify ζ_2 a bit later.)

Define $Y_1, Y_2, Y_3 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^{a+t-\alpha}-1} \zeta_1^{-i} \sigma^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p^\alpha-1} \zeta_2^{-i} \rho_1^i \cdot X_3.$$

It follows that

$$\sigma : Y_1 \mapsto \zeta_1 Y_1, \quad Y_2 \mapsto Y_2,$$

$$\rho_1 : Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_2 Y_2,$$

$$\rho : Y_1 \mapsto \xi^n Y_1, \quad Y_2 \mapsto \zeta_2^{-n} Y_2.$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H .

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2$ for $0 \leq i \leq p^b - 1$. We have now

$$\sigma : x_i \mapsto \zeta_1^{k^i} \xi^{nw_i} x_i, \quad y_i \mapsto \zeta_2^{-nw_i} y_i,$$

$$\tau : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^b-1} \mapsto x_0,$$

$$y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^b-1} \mapsto y_0,$$

$$\rho : x_i \mapsto \xi^n x_i, y_i \mapsto \zeta_2^{-n} y_i.$$

for $0 \leq i \leq p^b - 1$. We find that $Y = (\bigoplus_{0 \leq i \leq p^b-1} K \cdot x_i) \oplus (\bigoplus_{0 \leq i \leq p^b-1} K \cdot y_i)$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i : 0 \leq i \leq p^b - 1)^{\tilde{G}}$ is rational over K .

For $1 \leq i \leq p^b - 1$, define $U_i = x_i/x_{i-1}$ and $V_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \leq i \leq p^b - 1) = K(x_0, y_0, U_i, V_i : 1 \leq i \leq p^b - 1)$ and for every $g \in \tilde{G}$

$$g \cdot x_0 \in K(U_i, V_i) \cdot x_0, \quad g \cdot y_0 \in K(U_i, V_i) \cdot y_0,$$

while the subfield $K(U_i, V_i : 1 \leq i \leq p^b - 1)$ is invariant by the action of \tilde{G} , i.e.,

$$\begin{aligned} \sigma : U_i &\mapsto \zeta_1^{k^i - k^{i-1}} \xi^{nk^{i-1}} U_i, \quad V_i \mapsto \zeta_2^{-nk^{i-1}} V_i, \\ \tau : U_1 &\mapsto U_2 \mapsto \cdots \mapsto U_{p^b-1} \mapsto (U_1 \cdots U_{p^b-1})^{-1}, \\ V_1 &\mapsto V_2 \mapsto \cdots \mapsto V_{p^b-1} \mapsto (V_1 \cdots V_{p^b-1})^{-1}, \\ \rho : U_i &\mapsto U_i, \quad V_i \mapsto V_i. \end{aligned}$$

for $1 \leq i \leq p^b - 1$. From Theorem 2.2 it follows that if $K(U_i, V_i : 1 \leq i \leq p^b - 1)^{\tilde{G}}$ is rational over K , so is $K(x_i, y_i : 0 \leq i \leq p^b - 1)^{\tilde{G}}$ over K .

Since ρ acts trivially on $K(U_i, V_i)$, we find that $K(U_i, V_i)^{\tilde{G}} = K(U_i, V_i)^{\langle \sigma, \tau \rangle}$.

Subcase 3.a. Let $a - \alpha \leq r$. Thus we can write $r = a - \alpha + r_1$ for some $r_1 \geq 0$. Therefore, $\zeta_1^{k^{i-1}(k-1)} = \xi^{k^{i-1}p^{r_1}}$ for all i . Define $\zeta_2 = \xi^{(1+sp^{r_1})p^{t-\alpha}}$, a primitive p^α -th root of unity.

Define $v_i = U_i^{p^{t-\alpha}} V_i$. Since $\xi^{(n+p^{r_1})p^{t-\alpha}} = \xi^{n(1+sp^{r_1})p^{t-\alpha}} = \zeta_2^n$, we have

$$(5.6) \quad \begin{aligned} \sigma : U_i &\mapsto \xi^{k^{i-1}n(1+sp^{r_1})} U_i, \quad v_i \mapsto v_i, \\ \tau : U_1 &\mapsto U_2 \mapsto \cdots \mapsto U_{p^b-1} \mapsto (U_1 \cdots U_{p^b-1})^{-1}, \\ v_1 &\mapsto v_2 \mapsto \cdots \mapsto v_{p^b-1} \mapsto (v_1 \cdots v_{p^b-1})^{-1} \end{aligned}$$

for $1 \leq i \leq p^b - 1$. Compare Formula (5.6) with Formula (5.1). They look almost the same. Apply the proof of Case 1.

Subcase 3.b. Let $a - \alpha > r$. Let η be a primitive p^{r+t} -th root of unity such that $\xi^n = \eta^{p^r}$. Whence $\xi^{nk^{i-1}} = \eta^{k^i - k^{i-1}}$. Since $a + t - \alpha > r + t$, we get that $\zeta_1 \eta$ is a primitive $p^{a+t-\alpha}$ -th root of unity. Put $\mu = (\zeta_1 \eta)^{p^{a+t-\alpha-(\alpha+r)}}$, a primitive $p^{\alpha+r}$ -th root of

unity, where $\alpha + r < a \leq a + t - \alpha$. Now, put $\zeta_2 = \mu^{sp^r}$, a primitive p^α -th root of unity (recall that $ns \equiv 1 \pmod{p^t}$). Thus $\zeta_2^{nk^{i-1}} = \mu^{k^i - k^{i-1}}$ for any i .

Define $v_i = U_i^{p^{a+t-\alpha-(\alpha+r)}} V_i$ for $1 \leq i \leq p^b - 1$. It follows that

$$(5.7) \quad \begin{aligned} \sigma &: U_i \mapsto (\zeta_1 \eta)^{k^i - k^{i-1}} U_i, \quad v_i \mapsto v_i, \\ \tau &: U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^b-1} \mapsto (U_1 \cdots U_{p^b-1})^{-1}, \\ &v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^b-1} \mapsto (v_1 \cdots v_{p^b-1})^{-1} \end{aligned}$$

for $1 \leq i \leq p^b - 1$. Compare Formula (5.7) with Formula (5.3). They are the same. Apply the proof of Case 3, Step I.

6. PROOF OF COROLLARY 1.6

Let $C = C_{p^{\alpha_1}} \times \cdots \times C_{p^{\alpha_s}} \leq Z(\tilde{G})$. Denote again by σ and τ the preimages of the generators of G and by ρ_1, \dots, ρ_s the generators of C , i.e., $\rho_i^{p^{\alpha_i}} = 1$. Then $[\sigma, \tau] = \rho_1^{\beta_1} \cdots \rho_s^{\beta_s}$ for $\beta_i \geq 0$, and we can assume for abuse of notation that $p^t = \text{ord}(\rho_1^{\beta_1}) = \max\{\text{ord}(\rho_1^{\beta_1}), \dots, \text{ord}(\rho_s^{\beta_s})\}$. It follows that $\tilde{G}' \cap \langle \rho_2, \dots, \rho_s \rangle = \{1\}$. Now we can apply Theorem 2.5 reducing the rationality problem of $K(\tilde{G})$ over K to the rationality problem of $K(\tilde{G}/\langle \rho_2, \dots, \rho_k \rangle)$ over K . Put $\tilde{G}_1 = \tilde{G}/\langle \rho_2, \dots, \rho_k \rangle$ and let us find an upper bound for the exponent of \tilde{G}_1/\tilde{G}'_1 . We may assume that \tilde{G}_1 has the presentation

$$\tilde{G}_1 = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{sp^\alpha}, \tau^{p^b} = \sigma^{mp^c} \rho^{p^\beta}, \rho^{p^t} = 1, \tau^{-1} \sigma \tau = \sigma^k \rho, \rho - \text{central} \rangle,$$

where s, m are positive integers, $1 \leq s < p^t, 0 \leq m < p^a, \gcd(sm, p) = 1, 0 \leq \alpha, \beta \leq t$ and $k = \varepsilon + p^r$.

Computations show that we will obtain an upper bound for the exponent of \tilde{G}_1 in the following situation: $\sigma^{p^a} = \rho^{sp^\alpha}, \tau^{p^b} = \sigma^{p^c} \rho^{p^\beta}$ for $\alpha < t, c < a, \beta < t$. We have $\tau^{p^{b+a-c}} = \rho^{p^\alpha + p^{\beta+a-c}}$. There are several possibilities for the order of τ .

Case 1. Let $\beta + a - c \geq t$. Then $\text{ord}(\tau) = p^{b+a+t-c-\alpha}$.

Case 2. Let $\beta + a - c < t$ and $\alpha < \beta + a - c$. Here again $\text{ord}(\tau) = p^{b+a+t-c-\alpha}$.

Case 3. Let $\beta + a - c < t$ and $\alpha \geq \beta + a - c$. We have $\text{ord}(\tau) = p^{b+t-\beta}$.

Note also that $\text{ord}(\sigma) = p^{a+t-\alpha} \leq p^{a+t} \leq p^{a+b+t-c}$.

Therefore, $\exp(\tilde{G}_1/\tilde{G}'_1) \leq \exp(\tilde{G}_1) \leq p^{a+b+t-c}$.

7. PROOF OF THEOREM 1.7

The case (i) follows from Theorem 2.3.

(ii) We will divide the proof into two steps.

Step I. Assume that \tilde{G} has the following presentation:

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{p^\alpha}, \tau^{p^b} = \sigma^{p^c} \rho^{p^\beta}, \rho^{p^t} = 1, \tau^{-1} \sigma \tau = \sigma^k \rho, \rho - \text{central} \rangle,$$

where $a \geq t, b \geq t, r \geq t, 0 \leq \alpha, \beta \leq t$ and $k = \varepsilon + p^r$. Note that for any i we have the relation $\tau^{-i} \sigma \tau^i = \sigma^{k^i} \rho^i$, since $\rho^k = \rho$.

Case 1. $\tilde{G} = G_1$, where G_1 is the group in Section 3. From Proposition 3.3 it follows that \tilde{G}' is cyclic and is generated by $\sigma^{p^r} \rho$.

Subcase 1.a. Let $a \geq r + t$. Then $(\sigma^{p^r} \rho)^{p^{a-r}} = \rho^{p^{a-r}} = 1$, so $\tilde{G}' \cap \langle \rho \rangle = 1$. Theorem 2.5 then implies that we can reduce the rationality problem of $K(\tilde{G})$ to $K(\tilde{G}/\langle \rho \rangle)$ over K , where $\tilde{G}/\langle \rho \rangle$ clearly is a metacyclic p -group.

Subcase 1.b. Let $a < r + t$. Then $\tau^{-p^t} \sigma \tau^{p^t} = \sigma^{k^{p^t}} = \sigma$, since $(1 + p^r)^{p^t} = 1 + A \cdot p^{r+t}$ for some integer A . This means that τ^{p^t} is in the center of \tilde{G} . Clearly, $\tilde{G}' \cap \langle \tau^{p^t} \rangle = 1$, so we can apply again theorem 2.5 reducing the rationality problem of $K(\tilde{G})$ to $K(\tilde{G}/\langle \tau^{p^t} \rangle)$ over K . In other words, we may assume that $\tau^{p^t} = 1$.

Let V be a K -vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in \tilde{G}} K \cdot x(g)$ where \tilde{G} acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in \tilde{G}$. Thus $K(V)^{\tilde{G}} = K(x(g) : g \in \tilde{G})^{\tilde{G}} = K(\tilde{G})$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_{i=0}^{p^a-1} x(\sigma^i), \quad X_2 = \sum_{i=0}^{p^t-1} x(\rho^i).$$

Note that $\sigma \cdot X_1 = X_1$ and $\rho \cdot X_2 = X_2$.

Let $\zeta = \zeta_{p^a} \in K$ be a primitive p^a -th root of unity and define $\xi = \zeta^{p^{a-t}}$. Thus ξ is a primitive p^t -th root of unity. Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^t-1} \xi^{-i} \rho^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p^a-1} \zeta^{-i} \sigma^i \cdot X_2.$$

It follows that

$$\sigma : Y_1 \mapsto Y_1, Y_2 \mapsto \zeta Y_2,$$

$$\rho : Y_1 \mapsto \xi Y_1, Y_2 \mapsto Y_2.$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup $\langle \sigma, \rho \rangle$.

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2$ for $0 \leq i \leq p^t - 1$. We have now

$$\begin{aligned}\sigma & : x_i \mapsto \xi^i x_i, \quad y_i \mapsto \zeta^{k^i} y_i \\ \tau & : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^t-1} \mapsto x_0, \\ & \quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^t-1} \mapsto y_0, \\ \rho & : x_i \mapsto \xi x_i, \quad y_i \mapsto y_i,\end{aligned}$$

for $0 \leq i \leq p^t - 1$.

We find that $Y = (\bigoplus_{0 \leq i \leq p^t-1} K \cdot x_i) \oplus (\bigoplus_{0 \leq i \leq p^t-1} K \cdot y_i)$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ is rational over K .

For $1 \leq i \leq p^t - 1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \leq i \leq p^t - 1) = K(x_0, y_0, u_i, v_i : 1 \leq i \leq p^t - 1)$ and for every $g \in \tilde{G}$

$$g \cdot x_0 \in K(u_i, v_i : 1 \leq i \leq p^t - 1) \cdot x_0, \quad g \cdot y_0 \in K(u_i, v_i : 1 \leq i \leq p^t - 1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \leq i \leq p^t - 1)$ is invariant by the action of \tilde{G} , i.e.,

$$\begin{aligned}\sigma & : u_i \mapsto \xi u_i, \quad v_i \mapsto \zeta^{k^i - k^{i-1}} v_i \\ \tau & : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^t-1} \mapsto (u_1 u_2 \cdots u_{p^t-1})^{-1}, \\ & \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^t-1} \mapsto (v_1 v_2 \cdots v_{p^t-1})^{-1}, \\ \rho & : u_i \mapsto u_i, \quad v_i \mapsto v_i,\end{aligned}$$

for $1 \leq i \leq p^t - 1$. From Theorem 2.2 it follows that if $K(u_i, v_i : 1 \leq i \leq p^t - 1)^{\tilde{G}}$ is rational over K , so is $K(x_i, y_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ over K .

Since ρ acts trivially on $K(u_i, v_i : 1 \leq i \leq p^t - 1)$, we find that $K(u_i, v_i : 1 \leq i \leq p^t - 1)^{\tilde{G}} = K(u_i, v_i : 1 \leq i \leq p^t - 1)^{\langle \sigma, \tau \rangle}$.

Recall that $a < r + t$. So we can write $r = a - t + r_1$ for some $r_1 \geq 1$. Since $\xi^k = \xi$, we have $\zeta^{k^{i-1}(k-1)} = \zeta^{k^{i-1}p^{a-t+r_1}} = \xi^{p^{r_1}}$ for $1 \leq i \leq p^t - 1$.

Define $w_i = v_i/(u_i)^{p^{r_1}}$. We have now that $K(u_i, v_i : 1 \leq i \leq p^t - 1) = K(u_i, w_i : 1 \leq i \leq p^t - 1)$ and

$$\begin{aligned}\sigma & : u_i \mapsto \xi u_i, \quad w_i \mapsto w_i \\ \tau & : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^t-1} \mapsto (u_1 u_2 \cdots u_{p^t-1})^{-1},\end{aligned}$$

$$w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{p^t-1} \mapsto (w_1 w_2 \cdots w_{p^t-1})^{-1}.$$

According to Lemma 2.8, we can linearize the action of τ on w_1, \dots, w_{p^t-1} .

Write $L = K(u_i : 1 \leq i \leq p^t - 1)$ and consider $L(w_i : 0 \leq i \leq p^t - 1)^{\langle \sigma, \tau \rangle}$. Note that the group $\langle \sigma, \tau \rangle$ acts on the field $L(w_i)$ as $\langle \sigma, \tau \rangle / \langle \sigma^{p^t} \rangle$ and is faithful on L . Thus we may apply Theorem 2.1 to $L(w_i : 1 \leq i \leq p^t - 1)^{\langle \sigma, \tau \rangle}$. It remains to show that $L^{\langle \sigma, \tau \rangle}$ is rational over K .

Define $z_1 = u_1^{p^t}$, $z_i = u_i / u_{i-1}$ for $2 \leq i \leq p^t - 1$. Then $L^{\langle \sigma \rangle} = K(z_i : 1 \leq i \leq p^t - 1)$ and the action of τ is given by

$$\begin{aligned} \tau : z_1 &\mapsto z_1 z_2^{p^t}, \\ z_2 &\mapsto z_3 \mapsto \cdots \mapsto z_{p^t-1} \mapsto (z_1 z_2^{p^t-1} z_3^{p^t-2} \cdots z_{p^t-1}^2)^{-1} \mapsto z_1 z_2^{p^t-2} z_3^{p^t-3} \cdots z_{p^t-2}^2 z_{p^t-1} \mapsto z_2. \end{aligned}$$

Define $s_1 = z_2$, $s_i = \tau^{i-1} \cdot z_2$ for $2 \leq i \leq p^t - 1$. Then $K(z_i : 1 \leq i \leq p^t - 1) = K(s_i : 1 \leq i \leq p^t - 1)$ and

$$\tau : s_1 \mapsto s_2 \mapsto \cdots \mapsto s_{p^t-1} \mapsto (s_1 s_2 \cdots s_{p^t-1})^{-1}.$$

The action of τ can be linearized according to Lemma 2.8. Thus $K(s_i : 1 \leq i \leq p^t - 1)^{\langle \tau \rangle}$ is rational over K by Theorem 1.1.

Case 2. $\tilde{G} = G_2$. We can apply the same argument of Subcase 1.a, so we will assume that $t + r > a$. Note that $\tau^{-p^t} \sigma \tau^{p^t} = \sigma^{(1+p^r)p^t} \rho^{p^t} = \sigma$, since $t + r > a$. Therefore, τ^{p^t} is in the center of \tilde{G} and the group $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ is abelian of order p^{a+b} . There are two possibilities for the decomposition of H as a direct product of cyclic groups.

Subcase 2.a. $b - \beta \geq t$. Define $\rho_1 = \tau^{p^{b-\beta}} \rho^{-1}$, $\rho_2 = \tau^{p^t}$. Then H is isomorphic to the direct product $\langle \sigma \rangle \times \langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\sigma^{p^a} = \rho_1^{p^\beta} = \rho_2^{p^{b-\beta}} = 1$. We have $\rho = \rho_1^{-1} \rho_2^{p^{b-\beta-t}}$.

Define $X_1, X_2, X_3 \in V^*$ by

$$X_1 = \sum_{i,j} x(\rho_1^i \rho_2^j), \quad X_2 = \sum_{i,j} x(\sigma^i \rho_1^j), \quad X_3 = \sum_{i,j} x(\sigma^i \rho_2^j).$$

Note that $\sigma \cdot X_2 = X_2$, $\sigma \cdot X_3 = X_3$, $\rho_1 \cdot X_1 = X_1$, $\rho_1 \cdot X_2 = X_2$, $\rho_2 \cdot X_1 = X_1$ and $\rho_2 \cdot X_3 = X_3$.

Let $\zeta = \zeta_{p^a} \in K$ be a primitive p^a -th root of unity. Define $\zeta_1 = \zeta^{p^{a-\beta}}$, a primitive p^β -th root of unity; $\zeta_2 = \zeta^{p^{a-b+\beta}}$, a primitive $p^{b-\beta}$ -th root of unity; and $\xi = \zeta^{p^{a-t}}$, a primitive p^t -th root of unity.

Define $Y_1, Y_2, Y_3 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^a-1} \zeta^{-i} \sigma^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p^\beta-1} \zeta_1^{-i} \rho_1^i \cdot X_3, \quad Y_3 = \sum_{i=0}^{p^{b-\beta}-1} \zeta_2^{-i} \rho_2^i \cdot X_2.$$

It follows that

$$\begin{aligned} \sigma &: Y_1 \mapsto \zeta Y_1, \quad Y_2 \mapsto Y_2, \quad Y_3 \mapsto Y_3, \\ \rho_1 &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_1 Y_2, \quad Y_3 \mapsto Y_3, \\ \rho_2 &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_2, \quad Y_3 \mapsto \zeta_2 Y_3, \\ \rho &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_1^{-1} Y_2, \quad Y_3 \mapsto \xi Y_3. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2 + K \cdot Y_3$ is a representation space of the subgroup H .

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2, z_i = \tau^i \cdot Y_3$ for $0 \leq i \leq p^t - 1$. We have now

$$\begin{aligned} \sigma &: x_i \mapsto \zeta^{k^i} x_i, \quad y_i \mapsto \zeta_1^{-i} y_i, \quad z_i \mapsto \xi^i z_i, \\ \tau &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^t-1} \mapsto x_0, \\ &: y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^t-1} \mapsto y_0, \\ &: z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{p^t-1} \mapsto \zeta_2 z_0, \\ \rho &: x_i \mapsto x_i, \quad y_i \mapsto \zeta_1^{-1} y_i, \quad z_i \mapsto \xi z_i, \end{aligned}$$

for $0 \leq i \leq p^t - 1$. We find that $Y = (\bigoplus_{0 \leq i \leq p^t-1} K \cdot x_i) \oplus (\bigoplus_{0 \leq i \leq p^t-1} K \cdot y_i) \oplus (\bigoplus_{0 \leq i \leq p^t-1} K \cdot z_i)$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ is rational over K .

Note that $\zeta_1 = \xi^{p^{t-\beta}}$. For $1 \leq i \leq p^t - 1$, define $U_i = x_i/x_{i-1}$ and $W_i = z_i/z_{i-1}$. For $0 \leq i \leq p^t - 1$, define $V_i = y_i z_i^{p^{t-\beta}}$. Thus $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1) = K(x_0, z_0, U_i, V_j, W_i : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ and for every $g \in \tilde{G}$

$$g \cdot x_0 \in K(U_i, V_j, W_i) \cdot x_0, \quad g \cdot z_0 \in K(U_i, V_j, W_i) \cdot z_0,$$

while the subfield $K(U_i, V_j, W_i : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ is invariant by the action of \tilde{G} , i.e.,

$$\begin{aligned} \sigma &: U_i \mapsto \zeta^{k^i - k^{i-1}} U_i, \quad V_j \mapsto V_j, \quad W_i \mapsto \xi W_i, \\ \tau &: U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^t-1} \mapsto (U_1 \cdots U_{p^t-1})^{-1}, \\ (7.1) \quad &: V_0 \mapsto V_1 \mapsto \cdots \mapsto V_{p^t-1} \mapsto \zeta_2^{p^{t-\beta}} V_0, \\ &: W_1 \mapsto W_2 \mapsto \cdots \mapsto W_{p^t-1} \mapsto \zeta_2 (W_1 \cdots W_{p^t-1})^{-1}, \end{aligned}$$

$$\rho : U_i \mapsto U_i, V_j \mapsto V_j, W_i \mapsto W_i,$$

for $1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1$. From Theorem 2.2 it follows that if $K(U_i, V_j, W_i : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)^{\tilde{G}}$ is rational over K , so is $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ over K .

Since ρ acts trivially on $K(U_i, V_j, W_i)$, we find that $K(U_i, V_j, W_i)^{\tilde{G}} = K(U_i, V_j, W_i)^{\langle \sigma, \tau \rangle}$.

Recall that we have $r = a - t + r_1$ for some $r_1 \geq 1$. Therefore, $\zeta^{k^i - k^{i-1}} = \xi^{p^{r_1}}$ for $0 \leq i \leq p^t - 1$. Let $\zeta_3 \in K$ be a primitive $p^{b+t-\beta}$ -th root of unity such that $\zeta = \zeta_3^{p^{b+t-\beta-a}}$. Then $\zeta_2 = \zeta^{p^{a-b+\beta}} = \zeta_3^{p^t}$.

For $1 \leq i \leq p^t - 1$ define $w_i = W_i / \zeta_3$ and define $u_i = U_i / w_i^{p^{r_1}}$. It follows that

$$\begin{aligned} \sigma : u_i &\mapsto u_i, V_j \mapsto V_j, w_i \mapsto \xi w_i, \\ \tau : u_1 &\mapsto u_2 \mapsto \cdots \mapsto u_{p^t-1} \mapsto (u_1 \cdots u_{p^t-1})^{-1}, \\ V_0 &\mapsto V_1 \mapsto \cdots \mapsto V_{p^t-1} \mapsto \zeta_2^{p^{t-\beta}} V_0, \\ w_1 &\mapsto w_2 \mapsto \cdots \mapsto w_{p^t-1} \mapsto (w_1 \cdots w_{p^t-1})^{-1}, \end{aligned}$$

for $1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1$.

Write $L = K(V_j, w_i : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ and consider $L(u_i : 1 \leq i \leq p^t - 1)^{\langle \sigma, \tau \rangle}$. Note that the group $\langle \sigma, \tau \rangle$ acts on the field $L(u_i)$ as $\langle \sigma, \tau \rangle / \langle \sigma^{p^t}, \tau^{p^b} \rangle$ and is faithful on L . According to Lemma 2.8, we can linearize the action of τ on u_1, \dots, u_{p^t-1} . Thus we may apply Theorem 2.1 to $L(u_i : 1 \leq i \leq p^t - 1)^{\langle \sigma, \tau \rangle}$. It remains to show that $L^{\langle \sigma, \tau \rangle}$ is rational over K .

Define $s_1 = w_1^{p^t}$, $s_j = w_j / w_{j-1}$ for $2 \leq j \leq p^t - 1$. We have $L^{\langle \sigma \rangle} = K(V_i, s_j : 0 \leq i \leq p^t - 1, 1 \leq j \leq p^t - 1)$. The action of τ is

$$(7.2) \quad \begin{aligned} \tau : V_0 &\mapsto V_1 \mapsto \cdots \mapsto V_{p^t-1} \mapsto \zeta_2^{p^{t-\beta}} V_0, \\ s_1 &\mapsto s_2^{p^t} s_1, s_2 \mapsto s_3 \mapsto \cdots \mapsto s_{p^t-1} \mapsto 1 / (s_1 s_2^{p^t-1} \cdots s_{p^t-1}^2). \end{aligned}$$

Define $t_1 = s_2, t_i = \tau^{i-1} \cdot s_2$ for $2 \leq i \leq p^t - 1$. Then $K(t_i, V_j) = K(s_i, V_j)$ and

$$\tau : t_1 \mapsto t_2 \mapsto \cdots \mapsto t_{p^t-1} \mapsto (t_1 t_2 \cdots t_{p^t-1})^{-1}.$$

The action of τ on $K(t_i : 1 \leq i \leq p^t - 1)$ can be linearized according to Lemma 2.8. Since τ acts faithfully on $K(V_j)$, we can apply Theorem 2.1. It remains to show that $K(V_j)^{\langle \tau \rangle}$ is rational over K .

Note that $\tau^{p^t} \cdot V_i = \zeta_2^{p^{t-\beta}} V_i$ for all i , where $\zeta_2^{p^{t-\beta}} = \zeta^{p^{a+t-b}}$ is a primitive p^{b-t} -th root of unity. Define $v_0 = V_0^{p^{b-t}}$, $v_i = V_i/V_{i-1}$ for $1 \leq i \leq p^t - 1$. Then $K(v_i) = K(V_i)^{\langle \tau^{p^t} \rangle}$ and we have

$$\tau : v_0 \mapsto v_0 v_1^{p^{b-t}}, \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^t-1} \mapsto \zeta_2^{p^{t-\beta}} (v_1 v_2 \cdots v_{p^t-1})^{-1}.$$

Put $\zeta_3 = \zeta^{p^{a-b}}$, a primitive p^b -th root of unity. Whence $\zeta_2^{p^{t-\beta}} = \zeta_3^{p^t}$. Define $r_i = v_i/\zeta_3$ for $1 \leq i \leq p^t - 1$. It follows that

$$\tau : r_1 \mapsto r_2 \mapsto \cdots \mapsto r_{p^t-1} \mapsto (r_1 r_2 \cdots r_{p^t-1})^{-1}.$$

Applying Lemma 2.8 once again, we are done.

Subcase 2.b. $b - \beta < t$. Define $\rho_1 = \tau^{p^t} \rho^{-p^{\beta-b+t}}$. Then H is isomorphic to the direct product $\langle \sigma \rangle \times \langle \rho \rangle \times \langle \rho_1 \rangle$, where $\sigma^{p^a} = \rho^{p^t} = \rho_1^{p^{b-t}} = 1$. We have $\tau^{p^t} = \rho_1 \rho^{p^{\beta-b+t}}$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_{i,j} x(\rho^i \rho_1^j), \quad X_2 = \sum_{i,j} x(\sigma^i \rho^j), \quad X_3 = \sum_{i,j} x(\sigma^i \rho_1^j).$$

Note that $\sigma \cdot X_2 = X_2$, $\sigma \cdot X_3 = X_3$, $\rho \cdot X_1 = X_1$, $\rho \cdot X_2 = X_2$, $\rho_1 \cdot X_1 = X_1$ and $\rho_1 \cdot X_3 = X_3$.

Let $\zeta = \zeta_{p^a} \in K$ be a primitive p^a -th root of unity. Define $\zeta_1 = \zeta^{p^{a-b+t}}$, a primitive p^{b-t} -th root of unity; and $\xi = \zeta^{p^{a-t}}$, a primitive p^t -th root of unity.

Define $Y_1, Y_2, Y_3 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^a-1} \zeta^{-i} \sigma^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p^t-1} \xi^{-i} \rho^i \cdot X_3, \quad Y_3 = \sum_{i=0}^{p^{b-t}-1} \zeta_1^{-i} \rho_1^i \cdot X_2.$$

It follows that

$$\sigma : Y_1 \mapsto \zeta Y_1, \quad Y_2 \mapsto Y_2, \quad Y_3 \mapsto Y_3,$$

$$\rho : Y_1 \mapsto Y_1, \quad Y_2 \mapsto \xi Y_2, \quad Y_3 \mapsto Y_3,$$

$$\rho_1 : Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_2, \quad Y_3 \mapsto \zeta_1 Y_3.$$

Thus $K \cdot Y_1 + K \cdot Y_2 + K \cdot Y_3$ is a representation space of the subgroup H .

Define $x_i = \tau^i \cdot Y_1$, $y_i = \tau^i \cdot Y_2$, $z_i = \tau^i \cdot Y_3$ for $0 \leq i \leq p^t - 1$. We have now

$$\sigma : x_i \mapsto \zeta^{k^i} x_i, \quad y_i \mapsto \xi^i y_i, \quad z_i \mapsto z_i,$$

$$\tau : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^t-1} \mapsto x_0,$$

$$y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^t-1} \mapsto \xi^{p^{\beta-b+t}} y_0,$$

$$z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{p^t-1} \mapsto \zeta_1 z_0,$$

$$\rho : x_i \mapsto x_i, y_i \mapsto \xi y_i, z_i \mapsto z_i,$$

for $0 \leq i \leq p^t - 1$. We find that $Y = (\bigoplus_{0 \leq i \leq p^t-1} K \cdot x_i) \oplus (\bigoplus_{0 \leq i \leq p^t-1} K \cdot y_i) \oplus (\bigoplus_{0 \leq i \leq p^t-1} K \cdot z_i)$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ is rational over K .

For $1 \leq i \leq p^t - 1$, define $U_i = x_i/x_{i-1}$ and $V_i = y_i/y_{i-1}$. Thus $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1) = K(x_0, y_0, U_i, V_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ and for every $g \in \tilde{G}$

$$g \cdot x_0 \in K(U_i, V_i, z_j) \cdot x_0, \quad g \cdot y_0 \in K(U_i, V_i, z_j) \cdot y_0,$$

while the subfield $K(U_i, V_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ is invariant by the action of \tilde{G} , i.e.,

$$\begin{aligned} \sigma : U_i &\mapsto \zeta^{k^i - k^{i-1}} U_i, \quad V_i \mapsto \xi V_i, \quad z_j \mapsto z_j, \\ \tau : U_1 &\mapsto U_2 \mapsto \cdots \mapsto U_{p^t-1} \mapsto (U_1 \cdots U_{p^t-1})^{-1}, \\ (7.3) \quad V_1 &\mapsto V_2 \mapsto \cdots \mapsto V_{p^t-1} \mapsto \xi^{p^{\beta-b+t}} (V_1 \cdots V_{p^t-1})^{-1}, \\ z_0 &\mapsto z_1 \mapsto \cdots \mapsto z_{p^t-1} \mapsto \zeta_1 z_0, \\ \rho : U_i &\mapsto U_i, \quad V_i \mapsto V_i, \quad z_j \mapsto z_j, \end{aligned}$$

for $1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1$. From Theorem 2.2 it follows that if $K(U_i, V_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)^{\tilde{G}}$ is rational over K , so is $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ over K .

Compare the actions of σ, τ, ρ in Formula (7.3) with those in Formula (7.1). They look almost the same. Let ζ_2 be a primitive $p^{b+t-\beta}$ -th root of unity such that $\zeta = \zeta_2^{p^{b+t-\beta-a}}$. This follows from the assumption that K contains a primitive root of unity of degree the exponent of \tilde{G} . Then $\zeta_2^{p^t} = \xi^{p^{\beta-b+t}}$. Define $v_i = V_i/\zeta_2$ and use the same method in Subcase 2.a.

Case 3. $\tilde{G} = G_3$. We have the relations $(\sigma^{p^{a-\alpha}} \rho^{-1})^{p^\alpha} = 1$ and $\tau^{-1} \sigma^{p^{a-\alpha}} \tau = \sigma^{p^{a-\alpha}} \rho^{p^{a-\alpha}}$. Clearly, $\tilde{G}' \cap \langle \sigma^{p^{a-\alpha}} \rho^{-1} \rangle = \{1\}$. If $a - \alpha \geq t$ then $\sigma^{p^{a-\alpha}} \in Z(\tilde{G})$. Theorem 2.5 then implies that we can reduce the rationality problem of $K(\tilde{G})$ to $K(\tilde{G}/\langle \sigma^{p^{a-\alpha}} \rho^{-1} \rangle)$ over K , where $\tilde{G}/\langle \sigma^{p^{a-\alpha}} \rho^{-1} \rangle$ clearly is a metacyclic p -group. Therefore, we will assume henceforth that $a - \alpha < t$.

We are going to show now that $\tau^{p^t} \in Z(\tilde{G})$. Indeed, we have

$$\tau^{-p^t} \sigma \tau^{p^t} = \sigma^{(1+p^r)p^t} = \sigma \cdot \sigma^{Ap^{r+t}} = \sigma \cdot \rho^{Ap^{\alpha+r+t-a}} = \sigma,$$

where A is some integer and $\alpha + r + t - a > t$ since $a < \alpha + t \leq \alpha + r$.

In this way, we obtain that the subgroup $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ is abelian and has an order p^{a+b} . Put $\rho_1 = \sigma^{p^{a-\alpha}} \rho^{-1}$ and $\rho_2 = \tau^{p^t}$. Then $H \cong \langle \sigma \rangle \times \langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\sigma^{p^{a+t-\alpha}} = \rho_1^{p^\alpha} = \rho_2^{p^{b-t}} = 1$. We have also $\rho = \sigma^{p^{a-\alpha}} \rho_1^{-1}$.

Define $X_1, X_2, X_3 \in V^*$ by

$$X_1 = \sum_{i,j} x(\rho_1^i \rho_2^j), \quad X_2 = \sum_{i,j} x(\sigma^i \rho_1^j), \quad X_3 = \sum_{i,j} x(\sigma^i \rho_2^j).$$

Note that $\sigma \cdot X_2 = X_2, \sigma \cdot X_3 = X_3, \rho_1 \cdot X_1 = X_1, \rho_1 \cdot X_2 = X_2, \rho_2 \cdot X_1 = X_1$ and $\rho_2 \cdot X_3 = X_3$.

Let $\zeta_1 \in K$ be a primitive $p^{a+t-\alpha}$ -th root of unity. Define $\zeta_3 = \zeta_1^{p^{a+2t-\alpha-b}}$, a primitive p^{b-t} -th root of unity; and $\xi = \zeta_1^{p^{a-\alpha}}$, a primitive p^t -th root of unity. Since $a - \alpha < t \leq r$, we can write $r = a - \alpha + r_1$ for some $r_1 > 0$. Define $\zeta_2 = \xi^{(1+p^{r_1})p^{t-\alpha}}$, a primitive p^α -th root of unity.

Define $Y_1, Y_2, Y_3 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^{a+t-\alpha}-1} \zeta_1^{-i} \sigma^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p^\alpha-1} \zeta_2^{-i} \rho_1^i \cdot X_3, \quad Y_3 = \sum_{i=0}^{p^{b-t}-1} \zeta_3^{-i} \rho_2^i \cdot X_2.$$

It follows that

$$\begin{aligned} \sigma &: Y_1 \mapsto \zeta_1 Y_1, \quad Y_2 \mapsto Y_2, \quad Y_3 \mapsto Y_3, \\ \rho_1 &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_2 Y_2, \quad Y_3 \mapsto Y_3, \\ \rho_2 &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_2, \quad Y_3 \mapsto \zeta_3 Y_3, \\ \rho &: Y_1 \mapsto \xi Y_1, \quad Y_2 \mapsto \zeta_2^{-1} Y_2, \quad Y_3 \mapsto Y_3. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2 + K \cdot Y_3$ is a representation space of the subgroup H .

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2, z_i = \tau^i \cdot Y_3$ for $0 \leq i \leq p^t - 1$. We have now

$$\begin{aligned} \sigma &: x_i \mapsto \zeta_1^{k^i} \xi^i x_i, \quad y_i \mapsto \zeta_2^{-i} y_i, \quad z_i \mapsto z_i, \\ \tau &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^t-1} \mapsto x_0, \\ &: y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^t-1} \mapsto y_0, \\ &: z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{p^t-1} \mapsto \zeta_3 z_0, \end{aligned}$$

$$\rho : x_i \mapsto \xi x_i, y_i \mapsto \zeta_2^{-1} y_i, z_i \mapsto z_i.$$

for $0 \leq i \leq p^t - 1$. We find that $Y = (\bigoplus_{0 \leq i \leq p^t - 1} K \cdot x_i) \oplus (\bigoplus_{0 \leq i \leq p^t - 1} K \cdot y_i) \oplus (\bigoplus_{0 \leq i \leq p^t - 1} K \cdot z_i)$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ is rational over K .

For $1 \leq i \leq p^t - 1$, define $U_i = x_i/x_{i-1}$ and $V_i = y_i/y_{i-1}$. Thus $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1) = K(x_0, y_0, U_i, V_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ and for every $g \in \tilde{G}$

$$g \cdot x_0 \in K(U_i, V_i, z_j) \cdot x_0, g \cdot y_0 \in K(U_i, V_i, z_j) \cdot y_0,$$

while the subfield $K(U_i, V_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ is invariant by the action of \tilde{G} , i.e.,

$$\begin{aligned} \sigma & : U_i \mapsto \zeta_1^{k^i - k^{i-1}} \xi U_i, V_i \mapsto \zeta_2^{-1} V_i, z_j \mapsto z_j, \\ \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^t - 1} \mapsto (U_1 \cdots U_{p^t - 1})^{-1}, \\ (7.4) \quad & V_1 \mapsto V_2 \mapsto \cdots \mapsto V_{p^t - 1} \mapsto (V_1 \cdots V_{p^t - 1})^{-1}, \\ & z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{p^t - 1} \mapsto \zeta_3 z_0, \\ \rho & : U_i \mapsto U_i, V_i \mapsto V_i, z_j \mapsto z_j. \end{aligned}$$

for $1 \leq i \leq p^t - 1$ and $0 \leq j \leq p^t - 1$. From Theorem 2.2 it follows that if $K(U_i, V_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)^{\tilde{G}}$ is rational over K , so is $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ over K .

Since ρ acts trivially on $K(U_i, V_i, z_j)$, we find that $K(U_i, V_i, z_j)^{\tilde{G}} = K(U_i, V_i, z_j)^{\langle \sigma, \tau \rangle}$.

Recall that $r_1 = r - a + \alpha > 0$. Therefore, $\zeta_1^{k-1} = \zeta_1^{p^{a-\alpha} + r_1} = \xi^{p^{r_1}}$ and also $\zeta_1^{k^{i-1}(k-1)} = \xi^{p^{r_1}}$ for all i .

Define $v_i = U_i^{p^{t-\alpha}} V_i$. Since $\zeta_2 = \xi^{(1+p^{r_1})p^{t-\alpha}}$, we have

$$\begin{aligned} \sigma & : U_i \mapsto \xi^{1+p^{r_1}} U_i, v_i \mapsto v_i, z_j \mapsto z_j, \\ \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^t - 1} \mapsto (U_1 \cdots U_{p^t - 1})^{-1}, \\ & v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^t - 1} \mapsto (v_1 \cdots v_{p^t - 1})^{-1}, \\ & z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{p^t - 1} \mapsto \zeta_3 z_0. \end{aligned}$$

for $1 \leq i \leq p^t - 1$ and $0 \leq j \leq p^t - 1$.

Write $L = K(U_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ and consider $L(v_i : 1 \leq i \leq p^t - 1)^{\langle \sigma, \tau \rangle}$. Note that the group $\langle \sigma, \tau \rangle$ acts on the field $L(v_i)$ as $\langle \sigma, \tau \rangle / \langle \sigma^{p^t} \rangle$ and is

faithful on L . According to Lemma 2.8, we can linearize the action of τ on v_1, \dots, v_{p^t-1} . Thus we may apply Theorem 2.1 to $L(v_i : 1 \leq i \leq p^t - 1)^{\langle \sigma, \tau \rangle}$. It remains to show that $L^{\langle \sigma, \tau \rangle}$ is rational over K .

Define $u_1 = U_1^{p^t}$, $u_i = U_i/U_{i-1}$ for $2 \leq i \leq p^t - 1$. Then $K(u_i, z_j, 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1) = L^{\langle \sigma \rangle}$ and the action of τ is

$$(7.5) \quad \begin{aligned} \tau : u_1 &\mapsto u_1 u_2^{p^t}, u_2 \mapsto u_3 \mapsto \dots \mapsto (u_1 u_2^{p^t-1} u_3^{p^t-2} \dots u_{p^t-1}^2)^{-1}, \\ z_0 &\mapsto z_1 \mapsto \dots \mapsto z_{p^t-1} \mapsto \zeta_3 z_0. \end{aligned}$$

Compare the action of τ in Formula (7.5) with that in Formula (7.2). Use the same method in Subcase 2.a.

Case 4. $\tilde{G} = G_4$. As in Case 3, we may assume that $a < \alpha + t$. Therefore, τ^{p^t} is in the center of \tilde{G} and the group $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ is abelian of order p^{a+b} . There are two possibilities for the decomposition of H as a direct product of cyclic groups.

Subcase 4.a. $b - \beta \leq t$. Define $\rho_1 = \sigma^{p^{a-\alpha}} \rho^{-1}$, $\rho_2 = \tau^{p^t} \rho^{-p^{\beta+t-b}}$. Then H is isomorphic to the direct product $\langle \sigma \rangle \times \langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\sigma^{p^{a+t-\alpha}} = \rho_1^{p^\alpha} = \rho_2^{p^{b-t}} = 1$. We have $\rho = \rho_1^{-1} \sigma^{p^{a-\alpha}}$ and $\tau^{p^t} = \rho_2 \rho^{p^{\beta+t-b}}$.

We can adopt exactly the same definitions from Case 3, starting with $X_1, X_2, X_3 \in V^*$ till the place where we define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2, z_i = \tau^i \cdot Y_3$ for $0 \leq i \leq p^t - 1$. Here only the action of τ is changed a little:

$$\begin{aligned} \sigma : x_i &\mapsto \zeta_1^{k^i} \xi^i x_i, y_i \mapsto \zeta_2^{-i} y_i, z_i \mapsto z_i, \\ \tau : x_0 &\mapsto x_1 \mapsto \dots \mapsto x_{p^t-1} \mapsto \xi^{p^{\beta+t-b}} x_0, \\ y_0 &\mapsto y_1 \mapsto \dots \mapsto y_{p^t-1} \mapsto \zeta_2^{-p^{\beta+t-b}} y_0, \\ z_0 &\mapsto z_1 \mapsto \dots \mapsto z_{p^t-1} \mapsto \zeta_3 z_0, \\ \rho : x_i &\mapsto \xi x_i, y_i \mapsto \zeta_2^{-1} y_i, z_i \mapsto z_i, \end{aligned}$$

for $0 \leq i \leq p^t - 1$.

Let $\zeta_4 \in K$ be a primitive $p^{b+t-\beta}$ -th root of unity such that $\zeta_4 = \zeta_1^{p^{a-\alpha-b+\beta}}$. Then $\zeta_4^{p^t} = \xi^{p^{\beta-b+t}}$. Similarly, we can define $p^{\alpha+b-\beta}$ -th root of unity ζ_5 such that $\zeta_5 = (\xi^{1+p^{r_1}})^{p^{t-(b-\beta+\alpha)}}$. Then $\zeta_2^{p^{\beta-b+t}} = \zeta_5^{p^t}$. Note that $\alpha + b - \beta < t + b - \beta$, so ζ_5 is contained in K .

Define $U_i = x_i/\zeta_4 x_{i-1}$ and $V_i = \zeta_5 y_i/y_{i-1}$ for $1 \leq i \leq p^t - 1$. Thus we obtain exactly the same actions in Case 3 given by Formula (7.4).

Subcase 4.b. $b - \beta > t$. We have $(\tau^{p^{b-\beta}} \rho^{-1})^{p^\beta} = 1$. Then $\tilde{G}' \cap \langle \tau^{p^{b-\beta}} \rho^{-1} \rangle = \{1\}$, so we can apply Theorem 2.5 reducing the rationality problem of $K(\tilde{G})$ to the rationality problem of $K(\tilde{G}_1)$ over K , where $\tilde{G}_1 \cong \tilde{G}/\langle \tau^{p^{b-\beta}} \rho^{-1} \rangle$. The group \tilde{G}_1 is generated by elements σ, τ and ρ such that $\sigma^{p^a} = \rho^{p^\alpha}$, $\tau^{p^{b-\beta}} = \rho$, $\rho^{p^t} = 1$, $[\sigma, \tau] = \rho$, ρ -central. Then the abelian subgroup $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ is of order $p^{a+b-\beta}$. Put $y = b + \alpha - \beta - t$.

Sub-subcase 4.b.a. Let $y \leq a$. Define $\rho_1 = \tau^{p^t} \sigma^{-p^{a-y}}$. Then H is isomorphic to the direct product $\langle \sigma \rangle \times \langle \rho_1 \rangle$, where $\sigma^{p^{a+t-\alpha}} = \rho_1^{p^y} = 1$. Note that $\tau^{p^t} = \rho_1 \sigma^{p^{a-y}}$ and $\rho = \sigma^{p^{a-\alpha}} \rho_1^{p^{y-\alpha}}$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_i x(\rho_1^i), \quad X_2 = \sum_i x(\sigma^i).$$

Note that $\sigma \cdot X_2 = X_2$ and $\rho_1 \cdot X_1 = X_1$.

Recall that $r_1 = r - a + \alpha > 0$. Let $\zeta_1 \in K$ be a primitive $p^{a+t-\alpha}$ -th root of unity, and put $\xi = \zeta_1^{p^{a-\alpha}}$, a primitive p^t -th root of unity. Then $\zeta_1^{k-1} = \zeta_1^{p^{a-\alpha}+r_1} = \xi^{p^{r_1}}$ and also $\zeta_1^{k^{i-1}(k-1)} = \xi^{p^{r_1}}$ for all i . Define $\zeta_2 = \zeta_1^{(1+p^{r_1})p^{a+t-\alpha-y}}$, a primitive p^y -th root of unity.

Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^{a+t-\alpha}-1} \zeta_1^{-i} \sigma^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p^y-1} \zeta_2^{-i} \rho_1^i \cdot X_2.$$

It follows that

$$\sigma : Y_1 \mapsto \zeta_1 Y_1, \quad Y_2 \mapsto Y_2,$$

$$\rho_1 : Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_2 Y_2,$$

$$\rho : Y_1 \mapsto \xi Y_1, \quad Y_2 \mapsto \zeta_2^{p^{y-\alpha}} Y_2.$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H .

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2$ for $0 \leq i \leq p^t - 1$. We have now

$$\sigma : x_i \mapsto \zeta_1^{k^i} \xi^i x_i, \quad y_i \mapsto (\zeta_2^{p^{y-\alpha}})^i y_i,$$

$$\tau : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^t-1} \mapsto \zeta_1^{p^{a-y}} x_0,$$

$$y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^t-1} \mapsto \zeta_2 y_0,$$

$$\rho : x_i \mapsto \xi x_i, \quad y_i \mapsto \zeta_2^{p^{y-\alpha}} y_i,$$

for $0 \leq i \leq p^t - 1$.

For $1 \leq i \leq p^t - 1$ define $U_i = x_i/x_{i-1}, V_i = y_i/y_{i-1}$. It follows that

$$\begin{aligned} \sigma : U_i &\mapsto \xi^{1+p^{r_1}} U_i, \quad V_i \mapsto \zeta_2^{p^{y-\alpha}} V_i, \\ \tau : U_1 &\mapsto U_2 \mapsto \cdots \mapsto U_{p^t-1} \mapsto \zeta_1^{p^{a-y}} (U_1 \cdots U_{p^t-1})^{-1}, \\ V_1 &\mapsto V_2 \mapsto \cdots \mapsto V_{p^t-1} \mapsto \zeta_2 (V_1 \cdots V_{p^t-1})^{-1}, \\ \rho : U_i &\mapsto U_i, \quad V_i \mapsto V_i, \end{aligned}$$

for $1 \leq i \leq p^t - 1$.

Since ρ acts trivially on $K(U_i, V_i : 1 \leq i \leq p^t - 1)$, we find that $K(U_i, V_i)^{\tilde{G}} = K(U_i, V_i)^{\langle \sigma, \tau \rangle}$.

Note that $\zeta_2^{p^{y-\alpha}} = \xi^{(1+p^{r_1})p^{t-\alpha}}$. For $1 \leq i \leq p^t - 1$ define $v_i = U_i^{-p^{t-\alpha}} V_i$. We have

$$\begin{aligned} \sigma : U_i &\mapsto \xi^{1+p^{r_1}} U_i, \quad v_i \mapsto v_i, \\ \tau : U_1 &\mapsto U_2 \mapsto \cdots \mapsto U_{p^t-1} \mapsto \zeta_1^{p^{a-y}} (U_1 \cdots U_{p^t-1})^{-1}, \\ v_1 &\mapsto v_2 \mapsto \cdots \mapsto v_{p^t-1} \mapsto \zeta_1^{p^{r+t-y}} (v_1 \cdots v_{p^t-1})^{-1}, \end{aligned}$$

for $1 \leq i \leq p^t - 1$.

Let $\zeta_3 \in K$ be a primitive $p^{b+t-\beta}$ -th root of unity such that $\zeta_1 = \zeta_3^{p^{b+t-\beta-(a+t-\alpha)}}$. Then $\zeta_3^{p^t} = \zeta_1^{p^{a-y}}$. Define $u_1 = U_1^{p^t}$ and for $2 \leq i \leq p^t - 1$ define $u_i = U_i/(\zeta_3 U_{i-1})$.

From the inequality $a < \alpha + t$ we get $a - r - t < 0$. Whence $y - r_1 + t = b - \beta + t + a - r - t < b - \beta + t$. Let $\zeta_4 \in K$ be a primitive p^{y-r_1+t} -th root of unity such that $\zeta_4 = \zeta_3^{p^{b+t-\beta-(y-r_1+t)}}$. Then $\zeta_4^{p^t} = \zeta_1^{p^{r+t-y}}$.

For $1 \leq i \leq p^t - 1$ define $w_i = v_i/\zeta_4$. The actions of σ and τ are then given by

$$\begin{aligned} \sigma : u_i &\mapsto u_i, \quad w_i \mapsto w_i, \\ \tau : u_1 &\mapsto u_1 u_2^{p^t}, \quad u_2 \mapsto u_3 \mapsto \cdots \mapsto (u_1 u_2^{p^t-1} u_3^{p^t-2} \cdots u_{p^t-1}^2)^{-1}, \\ w_1 &\mapsto w_2 \mapsto \cdots \mapsto w_{p^t-1} \mapsto (w_1 \cdots w_{p^t-1})^{-1}, \end{aligned}$$

for $1 \leq i \leq p^t - 1$. Since σ acts trivially on $K(u_i, w_i : 1 \leq i \leq p^t - 1)$, we find that $K(u_i, w_i)^{\langle \sigma, \tau \rangle} = K(u_i, w_i)^{\langle \tau \rangle}$. As we did before, we can easily linearize the action of τ applying Lemma 2.8.

Sub-subcase 4.b.b. Let $y > a$. Put $u = y - a > 0$. Define $\rho_1 = \sigma\tau^{-p^{t+u}}$ and $\rho_2 = \tau^{p^t}$. Therefore, $H \cong \langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\rho_1^a = \rho_2^{p^{b-\beta}} = 1$, $\sigma = \rho_1\rho_2^{p^u}$ and $\rho = \rho_2^{p^{b-\beta-t}}$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_i x(\rho_1^i), \quad X_2 = \sum_i x(\rho_2^i).$$

Note that $\rho_1 \cdot X_1 = X_1$ and $\rho_2 \cdot X_2 = X_2$.

Let $\zeta_2 \in K$ be a primitive $p^{b-\beta}$ -th root of unity. Put $\zeta_1 = \zeta_2^{p^u}$, a primitive $p^{a+t-\alpha}$ -th root of unity; $\xi = \zeta_1^{p^{a-\alpha}}$, a primitive p^t -th root of unity; $\zeta_3 = \zeta_1^{p^{t-\alpha}}$, a primitive p^a -th root of unity.

Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^a-1} \zeta_3^{-i} \rho_1^i \cdot X_2, \quad Y_2 = \sum_{i=0}^{p^{b-\beta}-1} \zeta_2^{-i} \rho_2^i \cdot X_1.$$

It follows that

$$\begin{aligned} \rho_1 &: Y_1 \mapsto \zeta_3 Y_1, \quad Y_2 \mapsto Y_2, \\ \rho_2 &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_2 Y_2, \\ \sigma &: Y_1 \mapsto \zeta_3 Y_1, \quad Y_2 \mapsto \zeta_1 Y_2, \\ \rho &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto \xi Y_2. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H .

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2$ for $0 \leq i \leq p^t - 1$. We have now

$$\begin{aligned} \sigma &: x_i \mapsto \zeta_3^{k^i} x_i, \quad y_i \mapsto \zeta_1^{k^i} \xi^i y_i, \\ \tau &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^t-1} \mapsto x_0, \\ &: y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^t-1} \mapsto \zeta_2 y_0, \\ \rho &: x_i \mapsto x_i, \quad y_i \mapsto \xi y_i, \end{aligned}$$

for $0 \leq i \leq p^t - 1$. Define $U_i = x_i/x_{i-1}$ and $V_i = y_i/y_{i-1}$ for $1 \leq i \leq p^t - 1$. It follows that

$$\begin{aligned} \sigma &: U_i \mapsto \zeta_3^{k^i-k^{i-1}} U_i, \quad V_i \mapsto \zeta_1^{k^i-k^{i-1}} \xi V_i, \\ \tau &: U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^t-1} \mapsto (U_1 U_2 \cdots U_{p^t-1})^{-1}, \\ &: V_1 \mapsto V_2 \mapsto \cdots \mapsto V_{p^t-1} \mapsto \zeta_2 (V_1 V_2 \cdots V_{p^t-1})^{-1}, \\ \rho &: U_i \mapsto U_i, \quad V_i \mapsto V_i, \end{aligned}$$

for $0 \leq i \leq p^t - 1$. Since ρ acts trivially on $K(U_i, V_i : 1 \leq i \leq p^t - 1)$, we find that $K(U_i, V_i)^{\tilde{G}} = K(U_i, V_i)^{\langle \sigma, \tau \rangle}$.

Recall that $r_1 = r - a + \alpha > 0$ and that $\zeta_1^{k^{i-1}(k-1)} = \xi^{p^{r_1}} = \xi^{p^{r+\alpha-a}}$ for all i . We also have $\zeta_3^{k^{i-1}(k-1)} = \xi^{p^{r_1+t-\alpha}} = \xi^{p^{r+t-a}}$ for all i . Put $\eta = \xi^{1+p^{r+\alpha-a}}$. Since $\langle \eta \rangle = \langle \xi \rangle$, there exists $w \in \mathbb{Z}$ such that $\xi^{p^{r+t-a}} = \eta^w$.

For $1 \leq i \leq p^t - 1$ define $u_i = U_i V_i^{-w}$. It follows that

$$\begin{aligned} \sigma & : u_i \mapsto u_i, \quad V_i \mapsto \xi^{1+p^{r+\alpha-a}} V_i \\ \tau & : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^t-1} \mapsto \zeta_2^{-w} (u_1 u_2 \cdots u_{p^t-1})^{-1}, \\ & \quad V_1 \mapsto V_2 \mapsto \cdots \mapsto V_{p^t-1} \mapsto \zeta_2 (V_1 V_2 \cdots V_{p^t-1})^{-1}, \end{aligned}$$

for $1 \leq i \leq p^t - 1$. Similar actions of σ and τ have been considered many times in the previous cases, so we omit the details of the final stage of the proof.

Case 5. $\tilde{G} = G_5$. From Proposition 3.3 it follows that \tilde{G} is cyclic and is generated by $\sigma^{-2+2^r} \rho$. If $a \geq t + 1$ then $(\sigma^{-2+2^r} \rho)^{2^{a-1}} = 1$ and whence $\tilde{G} \cap \langle \rho \rangle = \{1\}$. Theorem 2.5 then implies that we can reduce the rationality problem of $K(\tilde{G})$ to $K(\tilde{G}/\langle \rho \rangle)$ over K , where $\tilde{G}/\langle \rho \rangle$ clearly is a metacyclic p -group.

Now, let $a = t$. Whence $r = t$, i.e., we have the relations $\sigma^{2^t} = \rho^{2^t} = \tau^{2^b} = 1$ and $\tau^{-1} \sigma \tau = \sigma^{-1} \rho$. If $b \geq t + 1$ we can apply the proof of Case 5 in Theorem 1.5, where we require a primitive 2^{t+1} -th root of unity in K . Therefore, we will assume that $\tau^{2^t} = 1$.

From $\tau^{-2} \sigma \tau^2 = \sigma$ it follows that τ^2 is in the center $Z(\tilde{G})$. Since $\tilde{G} \cap \langle \tau^2 \rangle = \{1\}$, we can apply Theorem 2.5 so that we reduce the rationality problem of $K(\tilde{G})$ to $K(\tilde{G}/\langle \tau^2 \rangle)$ over K . In this way, we can assume that $\tau^2 = 1$. Note that $\text{ord}(\tau^2) = 2^{t-1} < \text{ord}(\sigma) = 2^t = \exp(\tilde{G}_1/\tilde{G}'_1)$ for $\tilde{G}_1 = \tilde{G}/\langle \tau^2 \rangle$.

Let ξ be a primitive p^t -th root of unity. Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_{i=0}^{2^t-1} x(\sigma^i), \quad X_2 = \sum_{i=0}^{2^t-1} x(\rho^i).$$

Note that $\sigma \cdot X_1 = X_1$ and $\rho \cdot X_2 = X_2$.

Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{2^t-1} \xi^{-i} \rho^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{2^t-1} \xi^{-i} \sigma^i \cdot X_2.$$

It follows that

$$\begin{aligned}\sigma & : Y_1 \mapsto Y_1, Y_2 \mapsto \xi Y_2, \\ \rho & : Y_1 \mapsto \xi Y_1, Y_2 \mapsto Y_2.\end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup $\langle \sigma, \rho \rangle$.

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2$ for $0 \leq i \leq 1$. We have now

$$\begin{aligned}\sigma & : x_i \mapsto \xi^{\delta_i} x_i, y_i \mapsto \xi^{\gamma_i} y_i \\ \tau & : x_0 \mapsto x_1 \mapsto x_0, \\ & y_0 \mapsto y_1 \mapsto y_0, \\ \rho & : x_i \mapsto \xi x_i, y_i \mapsto y_i,\end{aligned}$$

where $\delta_i = 0, \gamma_i = 1$ for i -even; $\delta_i = 1, \gamma_i = -1$ for i -odd; and $0 \leq i \leq 1$.

We find that $Y = \bigoplus_{0 \leq i \leq 1} K \cdot x_i$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i : 0 \leq i \leq 1)^{\tilde{G}}$ is rational over K . The rationality of $K(x_i : 0 \leq i \leq 1)^{\tilde{G}}$ over K follows from Theorem 2.7.

Case 6. $\tilde{G} = G_6$. Similarly to Case 5, we may assume that $a = t$. Whence $r = t$, i.e., we have the relations $\sigma^{2^t} = \rho^{2^t} = 1, \tau^{2^b} = \rho^{2^b}$ and $\tau^{-1}\sigma\tau = \sigma^{-1}\rho$. Since in the proof of Case 6 in Theorem 1.5 we require a primitive 2^{t+1} -th root of unity in K , and $\text{ord}(\tau) \geq 2^{b+1} \geq 2^{t+1}$, we may apply the same proof.

Case 7. $\tilde{G} = G_7$. From the relation $\tau^{-1}\sigma^{2^a}\tau = \sigma^{-2^a+2^{a+r}} = \sigma^{2^a}$ it follows that the order of σ is 2^{a+1} . Therefore, $\alpha = t - 1$. We can apply the proof of Case 7 in Theorem 1.5, since there we require only primitive roots of unity of smaller order than the exponent of \tilde{G} .

Case 8. $\tilde{G} = G_8$. The proof is almost the same as in Case 7. The only difference is the action of τ .

Case 9. $\tilde{G} = G_9$. Recall that $c \geq r \geq t$. The proof is exactly the same as in Case 1. Indeed, if $a < r + t$ we have that τ^{p^t} is in the center of \tilde{G} and $\tilde{G}' \cap \langle \tau^{p^t} \rangle = 1$, so we can apply Theorem 2.5. Thus we may assume again that $\tau^{p^t} = 1$.

Case 10. $\tilde{G} = G_{10}$. Similarly to Case 2, we may assume that $a < t + r$ and $\tau^{p^t} \in Z(\tilde{G})$. The group $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ is abelian of order p^{a+b} .

Subcase 10.a. $b - \beta \geq t$.

Sub-subcase 10.a.a. $c + t \geq b$. Define $\rho_1 = \tau^{p^{b-\beta}} \sigma^{-p^{c-\beta}} \rho^{-1}$ and $\rho_2 = \tau^{p^t} \sigma^{-p^{c+t-b}}$. Then H is isomorphic to the direct product $\langle \sigma \rangle \times \langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\sigma^{p^a} = \rho_1^{p^\beta} = \rho_2^{p^{b-\beta}} = 1$. We have $\rho = \rho_1^{-1} \rho_2^{p^{b-\beta-t}}$ and $\tau^{p^t} = \rho_2 \sigma^{p^{c+t-b}}$.

The only difference now with Subcase 2.a is the action of τ on x_i 's. As we did many times so far, with a proper adjustment of the variables we can easily repair this 'defect'. A similar situation will reappear for all remaining cases, subcases and sub-subcases. For each subcase that follows, it is tiresome but not difficult to change the variables in such a way that we may directly apply the proof of the respective subcases of cases 2,3 and 4. As an illustration, we give some extra explanation in Subcase 11.b. For the remaining subcases we will write only the decompositions of $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ as a direct product of cyclic groups.

Sub-subcase 10.a.b. $c + t < b$ and $c + t < a + \beta$. We have $\tau^{p^{b+t-\beta}} = \sigma^{p^{c+t-\beta}} \neq 1$. If we suppose that a power of $\sigma^{p^c} \rho^{p^\beta}$ is in \tilde{G}' , we get $c = r + \beta$, whence $c + t - \beta = r + t > a$, a contradiction. Therefore, $\langle \tau^{p^t} \rangle \cap \tilde{G}' = \{1\}$ and we can apply Theorem 2.5, reducing this subcase to Case 1.

Sub-subcase 10.a.c. $c + t < b$ and $c + t \geq a + \beta$. We have $\tau^{p^{b+a-c}} = \rho^{p^{\beta+a-c}}$. Define $\rho_1 = \tau^{p^{b-\beta}} \sigma^{-p^{c-\beta}} \rho^{-1}$ and $\rho_2 = \tau^{p^t}$. Then H is isomorphic to the direct product $\langle \sigma \rangle \times \langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\sigma^{p^a} = \rho_1^{p^\beta} = \rho_2^{p^{b-\beta}} = 1$ and $\rho = \rho_1^{-1} \rho_2^{p^{b-\beta-t}} \sigma^{-p^{c-\beta}}$.

Subcase 10.b. $b - \beta < t$. Then $c \geq \beta > b - t$, i.e., $c - b + t > 0$. Define $\rho_1 = \tau^{p^t} \sigma^{-p^{c-b+t}} \rho^{-p^{\beta-b+t}}$. Then H is isomorphic to the direct product $\langle \sigma \rangle \times \langle \rho \rangle \times \langle \rho_1 \rangle$, where $\sigma^{p^a} = \rho^{p^t} = \rho_1^{p^{b-t}} = 1$. We have $\tau^{p^t} = \rho_1 \sigma^{p^{c+t-b}} \rho^{p^{\beta-b+t}}$.

Case 11. $\tilde{G} = G_{11}$. Similarly to Case 3, we may assume that $a < t + \alpha$ and $\tau^{p^t} \in Z(\tilde{G})$. The subgroup $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ is abelian and has an order p^{a+b} .

Subcase 11.a. $c - b + t \geq 0$. Put $\rho_1 = \sigma^{p^{a-\alpha}} \rho^{-1}$ and $\rho_2 = \tau^{p^t} \sigma^{-p^{c-b+t}}$. Then $H \cong \langle \sigma \rangle \times \langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\sigma^{p^{a+t-\alpha}} = \rho_1^{p^\alpha} = \rho_2^{p^{b-t}} = 1$. We have also $\rho = \sigma^{p^{a-\alpha}} \rho_1^{-1}$ and $\tau^{p^t} = \rho_2 \sigma^{p^{c-b+t}}$.

Subcase 11.b. $c - b + t < 0$. Put $\rho_1 = \sigma \tau^{-p^{b-c}}$, $\rho_2 = \sigma^{p^{a-\alpha}} \rho^{-1}$, $\rho_3 = \tau^{p^t}$. Then $H \cong \langle \rho_1 \rangle \times \langle \rho_2 \rangle \times \langle \rho_3 \rangle$, where $\rho_1^{p^c} = \rho_2^{p^\alpha} = \rho_3^{p^{a+b-c-\alpha}} = 1$. Note that $\sigma = \rho_1 \rho_3^{p^{b-c-t}}$ and $\rho = \rho_1^{p^{a-\alpha}} \rho_2^{-1} \rho_3^{p^{a-\alpha+b-c-t}}$. In this situation we need to adjust first the variables so that the actions of σ and ρ become the same as in Case 3.

Let $\zeta \in K$ be a primitive $p^{a+b-c-\alpha}$ -th root of unity. Define $\zeta_1 = \zeta^{p^{b-c-t}}$, a primitive $p^{a+t-\alpha}$ -th root of unity; $\zeta_2 = \zeta_1^{p^{a+t-\alpha-c}}$, a primitive p^c -th root of unity; and $\xi = \zeta_1^{p^{a-\alpha}}$, a primitive p^t -th root of unity. Since $a - \alpha < t \leq r$, we can write $r = a - \alpha + r_1$ for some $r_1 > 0$. Define $\zeta_3 = \xi^{(1+p^{r_1})p^{t-\alpha}}$, a primitive p^α -th root of unity.

Define $Y_1, Y_2, Y_3 \in V^*$ so that we have the actions

$$\begin{aligned} \rho_1 &: Y_1 \mapsto \zeta_2 Y_1, Y_2 \mapsto Y_2, Y_3 \mapsto Y_3, \\ \rho_2 &: Y_1 \mapsto Y_1, Y_2 \mapsto \zeta_3 Y_2, Y_3 \mapsto Y_3, \\ \rho_3 &: Y_1 \mapsto Y_1, Y_2 \mapsto Y_2, Y_3 \mapsto \zeta Y_3, \\ \rho &: Y_1 \mapsto \zeta_2^{p^{a-\alpha}} Y_1, Y_2 \mapsto \zeta_3^{-1} Y_2, Y_3 \mapsto \xi Y_3, \\ \sigma &: Y_1 \mapsto \zeta_2 Y_1, Y_2 \mapsto Y_2, Y_3 \mapsto \zeta_1 Y_3. \end{aligned}$$

Define $Z_1 = Y_3, Z_2 = Y_2, Z_3 = Y_3^{p^{a+t-\alpha-c}} Y_1^{-1}$. It is easily seen now that the actions of σ and ρ on $K(Z_1, Z_2, Z_3)$ are exactly the same as the actions of σ and ρ on $K(Y_1, Y_2, Y_3)$ in Case 3.

Case 12. $\tilde{G} = G_{12}$. We may again assume that $a < t + \alpha$ and $\tau^{p^t} \in Z(\tilde{G})$. The subgroup $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ is abelian and has an order p^{a+b} .

Subcase 12.a. $b - \beta \leq t$. Then $c \geq \beta \geq b - t$, i.e., $c - b + t \geq 0$. Define $\rho_1 = \sigma^{p^{a-\alpha}} \rho^{-1}, \rho_2 = \tau^{p^t} \sigma^{-p^{c+t-b}} \rho^{-p^{\beta+t-b}}$. Then H is isomorphic to the direct product $\langle \sigma \rangle \times \langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\sigma^{p^{a+t-\alpha}} = \rho_1^{p^\alpha} = \rho_2^{p^{b-t}} = 1$. We have $\rho = \rho_1^{-1} \sigma^{p^{a-\alpha}}$ and $\tau^{p^t} = \rho_2 \sigma^{p^{c+t-b}} \rho^{p^{\beta+t-b}}$.

Subcase 12.b. $b - \beta > t$. We have $(\tau^{p^{b-\beta}} \sigma^{-p^{c-\beta}} \rho^{-1})^{p^\beta} = 1$. Clearly, $\tilde{G}' \cap \langle \tau^{p^{b-\beta}} \sigma^{-p^{c-\beta}} \rho^{-1} \rangle = \{1\}$, so we can apply Theorem 2.5 reducing the rationality problem of $K(\tilde{G})$ to the rationality problem of $K(\tilde{G}_1)$ over K , where $\tilde{G}_1 \cong \tilde{G} / \langle \tau^{p^{b-\beta}} \sigma^{-p^{c-\beta}} \rho^{-1} \rangle$. The group \tilde{G}_1 is generated by elements σ, τ and ρ such that $\sigma^{p^a} = \rho^{p^\alpha}, \tau^{p^{b-\beta}} = \sigma^{p^{c-\beta}} \rho, \rho^{p^t} = 1, [\sigma, \tau] = \rho, \rho$ - central. Then the abelian subgroup $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ is of order $p^{a+b-\beta}$.

Sub-subcase 12.b.a. Let $b - c - t > 0$ and $\alpha \geq \beta + a - c$. Put $c - \beta + \alpha = a + u$, where $u \geq 0$. Define $\rho_1 = \sigma^{1+p^u} \tau^{-p^{b-c+u}}$ and $\rho_2 = \tau^{p^t}$. Then H is isomorphic to the direct product $\langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\rho_1^{p^a} = \rho_2^{p^{b-\beta}} = 1$. Let x be an integer such that $(1 + p^u)x \equiv 1 \pmod{p^{a+t-\alpha}}$. Whence $\sigma = (\sigma^{1+p^u})^x = \rho_1^x \rho_2^{xp^{b-c-t+u}}$ and $\rho = \rho_1^{-xp^{c-\beta}} \rho_2^{p^{b-\beta-t-xp^{b-t-\beta+u}}}$.

Sub-subcase 12.b.b. Let $b - c - t > 0$ and $\alpha < \beta + a - c$. Put $\beta + a - c = \alpha + v$, where $v > 0$. Define $\rho_1 = \sigma^{1+p^v} \tau^{-p^{b-c}}$ and $\rho_2 = \tau^{p^t}$. Then H is isomorphic to the direct product $\langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\rho_1^{p^{a-v}} = \rho_2^{p^{b-\beta+v}} = 1$. Let x be an integer such that $(1+p^v)x \equiv 1 \pmod{p^{a+t-\alpha}}$. Whence $\sigma = (\sigma^{1+p^v})^x = \rho_1^x \rho_2^{xp^{b-c-t}}$ and $\rho = \rho_1^{-xp^{c-\beta}} \rho_2^{p^{b-\beta-t} - xp^{b-t-\beta}}$.

Sub-subcase 12.b.c. Let $b - c - t \leq 0$ and $y = b + \alpha - \beta - t > a$. Put $y = a + u$ and $c + t = b + v$, where $u > 0, v \geq 0$. Note that $c + \alpha - \beta \geq b - t + \alpha - \beta = y > a$. Define $\rho_2 = \tau^{p^t} \sigma^{-p^v}$ and $\rho_1 = \sigma^{1+p^{c+\alpha-\beta-a}} \tau^{-p^{t+u}} = \sigma \rho_2^{-p^u}$. Then H is isomorphic to the direct product $\langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\rho_1^{p^a} = \rho_2^{p^{b-\beta}} = 1$. Note that $\sigma = \rho_1 \rho_2^u, \tau^{p^t} = \rho_1^v \rho_2^{1+p^{u+v}}$ and $\rho = \rho_2^{p^{b-\beta-t}}$.

Sub-subcase 12.b.d. Let $b - c - t \leq 0$ and $y = b + \alpha - \beta - t \leq a$. Put $c + t = b + v$, where $v \geq 0$. Define $\rho_1 = \tau^{p^t} \sigma^{-p^v} \sigma^{-p^{a-y}}$. Then H is isomorphic to the direct product $\langle \sigma \rangle \times \langle \rho_1 \rangle$, where $\sigma^{p^{a+t-\alpha}} = \rho_1^{p^y} = 1$. Note that $\tau^{p^t} = \rho_1 \sigma^{p^v+p^{a-y}}$ and $\rho = \rho_1^{p^{b-\beta-t}} \sigma^{p^{a-\alpha}}$.

Remark. For Sub-subcases 12.b.a and 12.b.b we have $\exp(\tilde{G}) \geq b > c + t \geq r + t$, so we can also apply Theorem 1.5.

Step II. Assume that \tilde{G} has the following general presentation:

$$\tilde{G} = \langle \sigma, \tau, \rho : \sigma^{p^a} = \rho^{sp^\alpha}, \tau^{p^b} = \sigma^{mp^c} \rho^{p^\beta}, \rho^{p^t} = 1, \tau^{-1} \sigma \tau = \sigma^k \rho, \rho - \text{central} \rangle,$$

where s, m are positive integers, $1 \leq s < p^t, 0 \leq m < p^a, \gcd(sm, p) = 1, 0 \leq \alpha, \beta \leq t$ and $k = \varepsilon + p^r$.

Here we have again 16 cases, which correspond to those in Step I. Since they all can be treated in an unified way, we will consider only Case 3.

Let $c = \beta = t$, i.e., $\tau^{p^b} = 1$.

We have the relations $(\sigma^{p^{a-\alpha}} \rho^{-s})^{p^\alpha} = 1$ and $\tau^{-1} \sigma^{p^{a-\alpha}} \tau = \sigma^{p^{a-\alpha}} \rho^{p^{a-\alpha}}$. Clearly, $\tilde{G}' \cap \langle \sigma^{p^{a-\alpha}} \rho^{-s} \rangle = \{1\}$. If $a - \alpha \geq t$ then $\sigma^{p^{a-\alpha}} \in Z(\tilde{G})$. Theorem 2.5 then implies that we can reduce the rationality problem of $K(\tilde{G})$ to $K(\tilde{G}/\langle \sigma^{p^{a-\alpha}} \rho^{-s} \rangle)$ over K , where $\tilde{G}_1 = \tilde{G}/\langle \sigma^{p^{a-\alpha}} \rho^{-s} \rangle$ is a metacyclic p -group. Indeed, if \tilde{G}_1 is generated by elements σ, τ, ρ such that $\sigma^{p^{a-\alpha}} = \rho^s$ and $\tau^{-1} \sigma \tau = \sigma^k \rho$, then $\tau^{-1} \sigma \tau = \sigma^{k+np^{a-\alpha}}$ for an integer n with $\rho = (\rho^s)^n$, i.e., $ns \equiv 1 \pmod{p^t}$.

Therefore, we will assume henceforth that $a - \alpha < t$. We are going to show now that $\tau^{p^t} \in Z(\tilde{G})$. Indeed, we have

$$\tau^{-p^t} \sigma \tau^{p^t} = \sigma^{(1+p^r)p^t} = \sigma \cdot \sigma^{Ap^{r+t}} = \sigma \cdot \rho^{Ap^{\alpha+r+t-a}} = \sigma,$$

where A is some integer and $\alpha + r + t - a > t$ since $a < \alpha + t \leq \alpha + r$.

In this way, we obtain that the subgroup $H = \langle \sigma, \rho, \tau^{p^t} \rangle$ is abelian and has an order p^{a+b} . Put $\rho_1 = \sigma^{p^{a-\alpha}} \rho^{-s}$ and $\rho_2 = \tau^{p^t}$. Then $H \cong \langle \sigma \rangle \times \langle \rho_1 \rangle \times \langle \rho_2 \rangle$, where $\sigma^{p^{a+t-\alpha}} = \rho_1^{p^\alpha} = \rho_2^{p^{b-t}} = 1$. Note that $\rho^s = \sigma^{p^{a-\alpha}} \rho_1^{-1}$, so $\rho = \sigma^{np^{a-\alpha}} \rho_1^{-n}$ with $ns \equiv 1 \pmod{p^t}$.

Define $X_1, X_2, X_3 \in V^*$ by

$$X_1 = \sum_{i,j} x(\rho_1^i \rho_2^j), \quad X_2 = \sum_{i,j} x(\sigma^i \rho_1^j), \quad X_3 = \sum_{i,j} x(\sigma^i \rho_2^j).$$

Note that $\sigma \cdot X_2 = X_2, \sigma \cdot X_3 = X_3, \rho_1 \cdot X_1 = X_1, \rho_1 \cdot X_2 = X_2, \rho_2 \cdot X_1 = X_1$ and $\rho_2 \cdot X_3 = X_3$.

Let $\zeta_1 \in K$ be a primitive $p^{a+t-\alpha}$ -th root of unity. Define $\zeta_3 = \zeta_1^{p^{a+2t-\alpha-b}}$, a primitive p^{b-t} -th root of unity; and $\xi = \zeta_1^{p^{a-\alpha}}$, a primitive p^t -th root of unity. Since $a - \alpha < t \leq r$, we can write $r = a - \alpha + r_1$ for some $r_1 > 0$. Define $\zeta_2 = \xi^{(1+sp^{r_1})p^{t-\alpha}}$, a primitive p^α -th root of unity (recall that $ns \equiv 1 \pmod{p^t}$).

Define $Y_1, Y_2, Y_3 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^{a+t-\alpha}-1} \zeta_1^{-i} \sigma^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p^\alpha-1} \zeta_2^{-i} \rho_1^i \cdot X_3, \quad Y_3 = \sum_{i=0}^{p^{b-t}-1} \zeta_3^{-i} \rho_2^i \cdot X_2.$$

It follows that

$$\begin{aligned} \sigma &: Y_1 \mapsto \zeta_1 Y_1, \quad Y_2 \mapsto Y_2, \quad Y_3 \mapsto Y_3, \\ \rho_1 &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_2 Y_2, \quad Y_3 \mapsto Y_3, \\ \rho_2 &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_2, \quad Y_3 \mapsto \zeta_3 Y_3, \\ \rho &: Y_1 \mapsto \xi^n Y_1, \quad Y_2 \mapsto \zeta_2^{-n} Y_2, \quad Y_3 \mapsto Y_3. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2 + K \cdot Y_3$ is a representation space of the subgroup H .

Define $x_i = \tau^i \cdot Y_1, y_i = \tau^i \cdot Y_2, z_i = \tau^i \cdot Y_3$ for $0 \leq i \leq p^t - 1$. We have now

$$\begin{aligned} \sigma &: x_i \mapsto \zeta_1^{k_i} \xi^{ni} x_i, \quad y_i \mapsto \zeta_2^{-ni} y_i, \quad z_i \mapsto z_i, \\ \tau &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^t-1} \mapsto x_0, \\ &: y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^t-1} \mapsto y_0, \\ &: z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{p^t-1} \mapsto \zeta_3 z_0, \\ \rho &: x_i \mapsto \xi^n x_i, \quad y_i \mapsto \zeta_2^{-n} y_i, \quad z_i \mapsto z_i. \end{aligned}$$

for $0 \leq i \leq p^t - 1$. We find that $Y = (\bigoplus_{0 \leq i \leq p^t - 1} K \cdot x_i) \oplus (\bigoplus_{0 \leq i \leq p^t - 1} K \cdot y_i) \oplus (\bigoplus_{0 \leq i \leq p^t - 1} K \cdot z_i)$ is a faithful \tilde{G} -subspace of V^* . Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ is rational over K .

For $1 \leq i \leq p^t - 1$, define $U_i = x_i/x_{i-1}$ and $V_i = y_i/y_{i-1}$. Thus $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1) = K(x_0, y_0, U_i, V_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ and for every $g \in \tilde{G}$

$$g \cdot x_0 \in K(U_i, V_i, z_j) \cdot x_0, \quad g \cdot y_0 \in K(U_i, V_i, z_j) \cdot y_0,$$

while the subfield $K(U_i, V_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)$ is invariant by the action of \tilde{G} , i.e.,

$$\begin{aligned} \sigma & : U_i \mapsto \zeta_1^{k^i - k^{i-1}} \xi^n U_i, \quad V_i \mapsto \zeta_2^{-n} V_i, \quad z_j \mapsto z_j, \\ \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^t - 1} \mapsto (U_1 \cdots U_{p^t - 1})^{-1}, \\ & \quad V_1 \mapsto V_2 \mapsto \cdots \mapsto V_{p^t - 1} \mapsto (V_1 \cdots V_{p^t - 1})^{-1}, \\ & \quad z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{p^t - 1} \mapsto \zeta_3 z_0, \\ \rho & : U_i \mapsto U_i, \quad V_i \mapsto V_i, \quad z_j \mapsto z_j. \end{aligned}$$

for $1 \leq i \leq p^t - 1$ and $0 \leq j \leq p^t - 1$. From Theorem 2.2 it follows that if $K(U_i, V_i, z_j : 1 \leq i \leq p^t - 1, 0 \leq j \leq p^t - 1)^{\tilde{G}}$ is rational over K , so is $K(x_i, y_i, z_i : 0 \leq i \leq p^t - 1)^{\tilde{G}}$ over K .

Since ρ acts trivially on $K(U_i, V_i, z_j)$, we find that $K(U_i, V_i, z_j)^{\tilde{G}} = K(U_i, V_i, z_j)^{(\sigma, \tau)}$.

Recall that $r_1 = r - a + \alpha > 0$. Therefore, $\zeta_1^{k-1} = \zeta_1^{p^{a-\alpha+r_1}} = \xi^{p^{r_1}}$ and also $\zeta_1^{k^{i-1}(k-1)} = \xi^{p^{r_1}}$ for all i .

Define $v_i = U_i^{p^{t-\alpha}} V_i$. Since $\xi^{(n+p^{r_1})p^{t-\alpha}} = \xi^{n(1+sp^{r_1})p^{t-\alpha}} = \zeta_2^n$, we have

$$\begin{aligned} \sigma & : U_i \mapsto \xi^{n+p^{r_1}} U_i, \quad v_i \mapsto v_i, \quad z_j \mapsto z_j, \\ \tau & : U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^t - 1} \mapsto (U_1 \cdots U_{p^t - 1})^{-1}, \\ & \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^t - 1} \mapsto (v_1 \cdots v_{p^t - 1})^{-1}, \\ & \quad z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{p^t - 1} \mapsto \zeta_3 z_0. \end{aligned}$$

for $1 \leq i \leq p^t - 1$ and $0 \leq j \leq p^t - 1$. The actions of σ and τ now are very similar to the actions in Case 3, Step I. Apply the same proof.

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